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Homework #6

Due May 3 (Wednesday) before class

This homework is not required for submission, but you can use it to replace your lowest homework grade.

Problem 1: Self-concordant Function

Exercise 1.1 (Logarithmic of Concave Quadratic Function) Let $f(x) = -\ln(\phi(x))$ with $\text{dom}(f) = \{x : \phi(x) > 0\}$, where

$$\phi(x) = -\frac{1}{2}x^T Qx + q^T x + c$$

where $Q \in \mathcal{S}^n_+$. Show that $f(x)$ is standard self-concordant.

Exercise 1.2 (Sum of Self-concordant Functions) Show that if $f_1(x)$ and $f_2(x)$ are self-concordant with constant κ_1, κ_2, then the function $f(x) = f_1(x)+f_2(x)$ is self-concordant with constant $\kappa = \max\{\kappa_1, \kappa_2\}$.

Problem 2: Self-concordant Barrier

Exercise 2.1 (Self-concordant Barrier for Quadratic Constraints) Prove that the logarithmic of concave quadratic function in Exercise 1.1

$$F(x) = -\ln\left(-\frac{1}{2}x^T Qx + q^T x + c\right)$$

is a 1-self-concordant barrier for the convex quadratic constraint set $X := \{x : \frac{1}{2}x^T Qx - q^T x - c \leq 0\}$.

Exercise 2.2 (Self-concordant Barrier for Lorentz Cone) Show that the function

$$F(x) = -\ln(x_n^2 - x_1^2 - \ldots - x_{n-1}^2)$$

is a 1-self-concordant barrier barrier for the Lorentz cone $X = \{x : x_n \geq \sqrt{x_1^2 + \ldots + x_{n-1}^2}\}$.

Exercise 2.3 (Scaling of Self-concordant Function) Show that if F_0 is a ν-self-concordant barrier, then $F(x) = \beta F_0(x)$ with $\beta \geq 1$ is a $(\beta\nu)$-self-concordant barrier.

Problem 3: Primal-Dual Path Following

Exercise 3.1 Verify the dual problem and KKT conditions discussed in class for the barrier primal problem

$$\begin{aligned} \min \quad & \text{Tr}(CX) - \mu \ln(\text{Det}(X)) \\ \text{s.t.} \quad & \text{Tr}(A_i X) = b_i, \quad i = 1, ..., m \end{aligned}$$

Problem 4: Optimality Condition as Fixed Point

Consider the convex minimization problem

$$\min_{x \in X} \quad f(x)$$

where $f(x)$ is convex and differentiable, X is convex closed. Show that the optimal solution is a fixed point of $x = \Pi_X(x - \gamma\nabla f(x))$ for any $\gamma > 0$, where $\Pi_X(\cdot)$ is the projection operator. Namely,

$$x^* \text{ is optimal} \Longleftrightarrow x^* = \Pi_X(x^* - \gamma\nabla f(x^*)), \forall \gamma > 0.$$