Homework #4
Due April 5 (Wednesday) at the beginning of class
Please show all work and intermediate steps. Late submission will lead to 0 credit.

Problem 1: Power Cone

Besides the second order cone and positive semidefinite cone, power cone is also very popular and can be used to model many convex programs. For a given $\alpha = (\alpha_1, \ldots, \alpha_m) > 0$ with $\sum_{i=1}^{m} \alpha_i = 1$, the power cone is defined as

$$K_\alpha = \{(x,y) \in \mathbb{R}_+^m \times \mathbb{R} : |y| \leq x_1^{\alpha_1} \cdots x_m^{\alpha_m}\}.$$

Exercise 1.1 (Cone) Show that $K_\alpha$ is a cone.

Exercise 1.2 (Dual Cone) Follow the steps below to show that the dual cone of power cone is

$$K^*_\alpha = P_\alpha := \{(u,v) \in \mathbb{R}_+^m \times \mathbb{R} : |v| \leq \left(\frac{u_1}{\alpha_1}\right)^{\alpha_1} \cdots \left(\frac{u_m}{\alpha_m}\right)^{\alpha_m}\}.$$

(a) Use Jensen’s inequality to prove the arithmetic-geometric mean inequality:

$$\sum_{i=1}^{m} \alpha_i y_i \geq \prod_{i=1}^{m} y_i^{\alpha_i}, \quad \forall y \in \mathbb{R}_+^m.$$

(b) Use the above inequality to show that $P_\alpha \subseteq K^*_\alpha$.

(c) Show that $P_\alpha \supseteq K^*_\alpha$.

Hint: consider the particular point $(\bar{x}, \bar{y})$ with $\bar{x}_i = \frac{u_i}{\alpha_i}$ and $\bar{y} = -\text{sgn}(v) \prod_{i=1}^{m} \left(\frac{\alpha_i}{u_i}\right)^{\alpha_i}$.

Problem 2: SOCP Reformulations

Exercise 2.1 (Log-Chebyshev Problem) In HW 1, we have shown that the following optimization problem

$$\min_x f(x) := \max_{k=1,\ldots,n} |\log(a_k^T x) - \log(b_k)|$$

s.t. $0 \leq x_i \leq 1, i = 1, \ldots, m$

where $a_k \in \mathbb{R}^m, b_k \in \mathbb{R}, k = 1, \ldots, n$ are given, is equivalent to the following convex optimization problem

$$\min_x \max_{k=1,\ldots,n} h(a_k^T x/b_k)$$

s.t. $0 \leq x_i \leq 1, i = 1, \ldots, m$

where $h(u) = \max(u, 1/u)$ for $u > 0$. Now reformulate the above problem into an SOCP. Use the fact that any hyperbolic constraints

$$z^2 \leq xy, x \geq 0, y \geq 0$$

can be rewritten as an second order conic constraint

$$\left\| \begin{bmatrix} 2z \\ x - y \end{bmatrix} \right\|_2 \leq x + y.$$
Exercise 2.2 (Sparse Group Lasso)  Reformulate the sparse group lasso model as an SOCP:

\[
\min_w \left\{ \|Xw - y\|_2^2 + \lambda \sum_{i=1}^p \|w_i\|_2 \right\}
\]

where \(X \in \mathbb{R}^{m \times n}, y \in \mathbb{R}^m\) are the input data, \(w = [w_1; \ldots; w_p]\) is the decision variable, with block \(w_i \in \mathbb{R}^{n_i}\) and \(n_1 + \ldots + n_p = n\).

Problem 3: SDP Reformulations

Exercise 3.1 (Spectral Norm Minimization)  The spectral norm of a general matrix \(X \in \mathbb{R}^{m \times n}\) is defined as the largest singular value of \(X\), i.e. the square root of the largest eigenvalue of the positive-semidefinite matrix \(X^T X\):

\[
\|X\| := \sigma_{\text{max}}(X) = \lambda_{\text{max}}(X^T X)
\]

Reformulate the spectral norm minimization problem below as an SDP,

\[
\min_{x \in \mathbb{R}^p} \sigma_{\text{max}} \left( \sum_{i=1}^p x_i A_i - B \right)
\]

where \(A_1, \ldots, A_p, B \in \mathbb{R}^{m \times n}\).

Hint: Use Schur complement lemma.

Exercise 3.2 (Inverse Matrix Minimization)  Reformulate the following minimization problem as an SDP:

\[
\min_x f(x) := \max_{1 \leq k \leq K} c_k^T A(x)^{-1} c_k
\]

where \(A(x) = \sum_{i=1}^p x_i A_i - B\), and \(A_1, \ldots, A_p, B \in S^n\). Assume the domain of the objective \(f\) is \(\text{dom}(f) = \{x \in \mathbb{R}^p : A(x) \succ 0\}\).

Problem 4: SDP Duality

We denote by \(S_r(A)\) the sum of the largest \(r\) eigenvalues of a symmetric matrix \(A \in S^n\) \((1 \leq r \leq n)\), i.e.,

\[
S_r(A) = \sum_{i=1}^r \lambda_i(A)
\]

where \(\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)\) are the eigenvalues of \(A\). In Lecture 15, we have shown that

\[
S_1(A) = \max_X \{\text{Tr}(AX) : \text{Tr}(X) = 1, \ X \succeq 0\}.
\]

Now use similar argument to show that

\[
S_k(A) = \max_X \{\text{Tr}(AX) : \text{Tr}(X) = r, \ 0 \preceq X \preceq I\}.
\]