Homework #4

Due April 5 (Wednesday) at the beginning of class
Please show all work and intermediate steps. Late submission will lead to 0 credit.

Problem 1: Power Cone

Besides the second order cone and positive semidefinite cone, power cone is also very popular and can be used to model many convex programs. For a given $\alpha = (\alpha_1, \dots, \alpha_m) > 0$ with $\sum_{i=1}^m \alpha_i = 1$, the power cone is defined as

$$K_\alpha = \{(x, y) \in \mathbf{R}_+^m \times \mathbf{R} : \ |y| \le x_1^{\alpha_1} \cdots x_m^{\alpha_m}\}.$$

Exercise 1.1 (Cone) Show that K_α is a cone.

Exercise 1.2 (Dual Cone) Follow the steps below to show that the dual cone of power cone is

$$K_\alpha^* = P_\alpha := \left\{ (u, v) \in \mathbf{R}_+^m \times \mathbf{R} : \ |v| \le \left(\frac{u_1}{\alpha_1}\right)^{\alpha_1} \cdots \left(\frac{u_m}{\alpha_m}\right)^{\alpha_m} \right\}.$$

(a) Use Jensen's inequality to prove the arithmetic-geometric mean inequality:

$$\sum_{i=1}^m \alpha_i y_i \ge \prod_{i=1}^m y_i^{\alpha_i}, \qquad \forall y \in \mathbf{R}_+^m.$$

(b) Use the above inequality to show that $P_\alpha \subseteq K_\alpha^*$.

(c) Show that $P_\alpha \supseteq K_\alpha^*$.
Hint: consider the particular point $(\bar{x}, \bar{y})$ *with* $\bar{x}_i = \frac{\alpha_i}{u_i}$ *and* $\bar{y} = -\text{sgn}(v) \prod_{i=1}^m \left(\frac{\alpha_i}{u_i}\right)^{\alpha_i}$.

Problem 2: SOCP Reformulations

Exercise 2.1 (Log-Chebyshev Problem) In HW 1, we have shown that the following optimization problem

$$\begin{aligned} \min_x \quad & f(x) := \max_{k=1,\dots,n} |\log(a_k^T x) - \log(b_k)| \\ \text{s.t.} \quad & 0 \le x_i \le 1, i = 1, \dots, m \end{aligned}$$

where $a_k \in \mathbf{R}^m, b_k \in \mathbf{R}, k = 1, \dots, n$ are given, is equivalent to the following convex optimization problem

$$\begin{aligned} \min_x \quad & \max_{k=1,\dots,n} h(a_k^T x / b_k) \\ \text{s.t.} \quad & 0 \le x_i \le 1, i = 1, \dots, m \end{aligned}$$

where $h(u) = \max(u, 1/u)$ for $u > 0$. Now reformulate the above problem into an SOCP.
Use the fact that any hyperbolic constraints

$$z^2 \le xy, x \ge 0, y \ge 0$$

can be rewritten as an second order conic constraint

$$\left\| \begin{bmatrix} 2z \\ x - y \end{bmatrix} \right\|_2 \le x + y.$$

Exercise 2.2 (Sparse Group Lasso) Reformulate the sparse group lasso model as an SOCP:

$$\min_{w} \left\{ \|Xw - y\|_2^2 + \lambda \sum_{i=1}^{p} \|w_i\|_2 \right\}$$

where $X \in \mathbf{R}^{m\times n}, y \in \mathbf{R}^m$ are the input data, $w = [w_1; \ldots; w_p]$ is the decision variable, with block $w_i \in \mathbf{R}^{n_i}$ and $n_1 + \ldots + n_p = n$.

Problem 3: SDP Reformulations

Exercise 3.1 (Spectral Norm Minimization) The spectral norm of a general matrix $X \in \mathbf{R}^{m\times n}$ is defined as the largest singular value of X, i.e. the square root of the largest eigenvalue of the positive-semidefinite matrix $X^T X$:

$$\|X\| := \sigma_{\max}(X) = \lambda_{\max}(X^T X)$$

Reformulate the spectral norm minimization problem below as an SDP,

$$\min_{x \in \mathbf{R}^p} \ \sigma_{\max}\Big(\sum_{i=1}^{p} x_i A_i - B \Big)$$

where $A_1, \ldots, A_p, B \in \mathbf{R}^{m\times n}$.
Hint: Use Schur complement lemma.

Exercise 3.2 (Inverse Matrix Minimization) Reformulate the following minimization problem as an SDP:

$$\min_{x} \ f(x) := \max_{1\le k\le K} c_k^T \mathcal{A}(x)^{-1} c_k$$

where $\mathcal{A}(x) = \sum_{i=1}^{p} x_i A_i - B$, and $A_1, \ldots, A_p, B \in S^n$. Assume the domain of the objective f is $\mathrm{dom}(f) = \{x \in \mathbf{R}^p : \mathcal{A}(x) \succ 0\}$.

Problem 4: SDP Duality

We denote by $S_r(A)$ the sum of the largest r eigenvalues of a symmetric matrix $A \in S^n$ $(1 \le r \le n)$, i.e.,

$$S_r(A) = \sum_{i=1}^{r} \lambda_i(A)$$

where $\lambda_1(A) \ge \lambda_2(A) \ge \cdots \ge \lambda_n(A)$ are the eigenvalues of A. In Lecture 15, we have shown that

$$S_1(A) = \max_{X} \{\mathrm{Tr}(AX) : \ \mathrm{Tr}(X) = 1, \ X \succeq 0\}.$$

Now use similar argument to show that

$$S_k(A) = \max_{X} \{\mathrm{Tr}(AX) : \ \mathrm{Tr}(X) = r, \ 0 \preceq X \preceq I\}.$$