Homework #2

Due Mar 1 (Wednesday) at the beginning of class
Please show all work and intermediate steps. Late submission will lead to 0 credit.

Problem 1: Subgradient and Subdifferential

Exercise 1.1 (Subdifferential) Calculate $\partial f(x)$ for the following functions

(a) $f(x) = \max(1, |x| - 1)$ on $\mathbf{R}$.

(b) $f(x) = \|x\|$, where $\|\cdot\|$ is a norm on $\mathbf{R}^n$.

Exercise 1.2 (Subdifferential of Pointwise Maximum) Let $f_1, \ldots, f_m$ be convex functions and $x \in \cap_{i=1}^m \text{int}(\text{dom}(f_i))$. Let $h(x) = \max_{1\le i\le m} f_i(x)$, and $I(x)$ be the set of all $i \in \{1, \ldots, m\}$ such that $f_i(x) = h(x)$.

(a) Use definition of subdifferential to show that $\text{Conv}(\{\partial f_i(x) : i \in I(x)\}) \subseteq \partial h(x)$.

(b) Use definiton of directional derivative to show that $h'(x; d) = \max_{i\in I(x)} f_i'(x; d)$.

(c) Use separation theorem to show that $\text{Conv}(\{\partial f_i(x) : i \in I(x)\}) \supseteq \partial h(x)$.
[*Hint: Use the fact that $\partial f_i(x)$ are compact convex and $h'(x; d) = \max_{g\in\partial h(x)} g^T d$.*]

Exercise 1.3 (Directional Derivative) Let f be a convex function and $x \in \text{dom}(f)$ and let d be such that $x + \alpha d \in \text{dom}(f)$ for $\alpha \in (0, \delta)$ for some $\delta > 0$. Show that the scalar function

$$\phi(\alpha) = \frac{f(x + \alpha d) - f(x)}{\alpha}$$

is non-decreasing function of α on $(0, \delta)$.

Problem 2: Convex Conjugate

Exercise 2.1 (Compute Conjugate) Calculate the conjugate of the following functions:

(a) $f(x) = e^x$ on $\mathbf{R}$

(b) $f(x) = \|x\|$ on $\mathbf{R}^n$

(c) $f(x) = \frac{1}{2}\|x\|^2$ on $\mathbf{R}^n$

(d) $f(x) = \log(\sum_{i=1}^n \exp\{x_i\})$ on $\mathbf{R}^n$

Exercise 2.2 (Calculus of Conjugate) Prove the following

(a) (Scalar Multiplication) Let $f(x)$ be convex and $\alpha > 0$, then

$$(\alpha f)^*(y) = \alpha f^*(y/\alpha)$$

(b) (Direct Summation) Let $f(x_1)$ and $g(x_2)$ be convex and $h(x_1, x_2) = f(x_1) + g(x_2)$, then

$$h^*(y_1, y_2) = f^*(y_1) + g^*(y_2)$$

(c) (Weighted Summation) Let $f(x)$ and $g(x)$ be closed convex functions,and $h(x) = f(x) + g(x)$, then

$$h^*(y) = \inf_z \{f^*(z) + g^*(y - z)\}$$

where the latter is the convolution of f^* and g^*.
[*Hint: First show that* $(\inf_z\{F(z) + G(y - z)\})^* = F^*(y) + G^*(y)$, and then apply with $F = f^*$, and $G = g^*$.]

Exercise 2.3 (Fenchel's Inequality) We already know that for any x and y, $x^T y \le f(x) + f^*(y)$. Show that $x^T y = f(x) + f^*(y)$ if and only if $y \in \partial f(x)$.

Problem 3: Lagrange Duality

Consider the following optimization problem in $\mathbf{R}^2$:

$$\begin{aligned} \min_{x_1,x_2} \quad & e^{-x_2} \\ \text{s.t.} \quad & \|x\|_2 \le x_1 \\ & x_2 \ge 0 \end{aligned}$$

Exercise 3.1 (Representation Issue) Through this example, we are going to see that duality gap is closely related to the "representation" of the constraints.

(a) Find the feasible set and optimal value.

(b) Let us write the problem as

$$\min\{e^{-x_2} : g(x) \le 0, x \in X\}$$

where $g(x) = \|x\|_2 - x_1$ and $X = \{(x_1, x_2) : x_2 \ge 0\}$. Does the slater condition holds? What is the dual optimal value? Is there a duality gap?

(c) Let us write the problem as

$$\min\{e^{-x_2} : g(x) \le 0, x \in X\}$$

where $g(x) = -x_2$ and $X = \{(x_1, x_2) : \|x\|_2 \le x_1\}$. Does the slater condition holds? What is the dual optimal value? Is there a duality gap?

Problem 4: Application in Finance

Consider assests $S_1, \ldots, S_n (n \ge 2)$ with random returns $\xi_1, \ldots \xi_n$. Let μ_i and σ_i denote the expected return and standard deviation of the random return of asset S_i, and ρ_{ij} denote the correlation coefficient of the returns of asset S_i and S_j. Denote $\mu = [\mu_1, \ldots, \mu_m]$ as the expected return of all assests, i.e. $\mathbf{E}[\xi] = \mu$. Denote $\Sigma = (\sigma_{ij})$ as the covariance matrix of the asset returns with $\sigma_{ii} = \sigma_i^2$ and $\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$ for $i \ne j$, i.e. $\mathbf{Var}(\xi) = \Sigma$.

Suppose an investor plans to invest the proportion x_i of his total funds in asset i, for $i = 1, \ldots, m$. The resulting portfolio is represented as $x = (x_1, \ldots, x_n)$ and $\sum_{i=1}^n x_i = 1$. Assume short sale is not allowed, i.e. $x_i \ge 0$ for any i. The investor wants to find the best portfolio strategy to maximize his total expected return and meanwhile minimize his "risk". One way to take both criterions into account is to minimize a linear combination

$$\lambda \mathbf{Var}[\xi^T x] - \mathbf{E}[\xi^T x]$$

where $\lambda > 0$ is a risk-aversion constant and balances the resturn and risk.

Exercise 4.1 Assume there are no restrictions on the portfolio. Formulate the above problem into an optimization model. Is the problem convex or not?

Exercise 4.2 Write down the Karush-Kuhn-Tucker optimality conditions for the problem.

Exercise 4.3 An alternative risk measure is Value-at-Risk(VaR) developed by financial engineers at J.P. Morgan. Given a probability level $\alpha \in (0, 1)$, the α-VaR of a random variable η is defined as:

$$\mathbf{VaR}_\alpha(\eta) := \min\{\gamma : P(\eta \ge \gamma) \le 1 - \alpha\}$$

Now change the objective to minimize the Value-at-Risk of the total return, i.e., $\mathbf{VaR}_\alpha(\xi^T x)$ with some $\alpha > 0.5$. Simplify the new optimization problem. Is the new problem convex or not? What if when ξ is a Gaussian random vector?

Exercise 4.4 A well-known modification of the Value-at-Risk is conditional Value-at-Risk (CVaR), which takes into account of the magnitute of random variables beyond the VaR value. Given a probability level $\alpha \in (0,1)$, the α-CVaR of a random variable η is defined as: $\mathbf{CVaR}_\alpha(\eta) := \mathbf{E}[\eta | \eta \geq \mathbf{VaR}_\alpha(\eta)]$. It can be shown that

$$\mathbf{CVaR}_\alpha(\eta) = \min_{\gamma > 0} \{\gamma + \frac{1}{1-\alpha}\mathbf{E}[(\eta - \gamma)_+]\}$$

where $u_+ := \max(u, 0)$. Now change the objective to minimize the conditional Value-at-Risk of the total return, i.e. $\mathbf{CVaR}_\alpha(\xi^T x)$ with some $\alpha > 0$. Simplify the new optimization problem. Is the new problem convex or not?