Homework #1

Due Feb 15 (Wednesday) at the beginning of class

Please show all work and intermediate steps. Late submission will lead to 0 credit.

Problem 1: Convex Sets

A real-valued function on $\mathbf{R}^n$ is called a *norm*, denoted as $\|\cdot\|$, if it satisfies the three properties:

- (*positivity*): $\forall x \in \mathbf{R}^n, \|x\| \geq 0$; $\|x\| = 0$ if an only if $x = 0$;
- (*homogeneity*): $\forall x \in \mathbf{R}^n, \alpha \in \mathbf{R}$: $\|\alpha x\| = |\alpha| \cdot \|x\|$;
- (*triangle inequality*): $\forall x, y \in \mathbf{R}^n : \|x + y\| \leq \|x\| + \|y\|$.

The standard norms on $\mathbf{R}^n$ are the ℓ_p-norms ($p \geq 1$):

$$\|x\|_p := \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$$

- [Euclidean norm]: $p = 2$, $\|x\|_2 := \sqrt{\sum_{i=1}^n x_i^2}$;
- [Manhattan norm]: $p = 1$, $\|x\|_1 := \sum_{i=1}^n |x_i|$;
- [Maximum norm]: $p = \infty$, $\|x\|_\infty := \max_{1\leq i\leq n} |x_i|$.

Exercise 1.1 (Dual Norm) The *dual norm* of $\|\cdot\|$ on $\mathbf{R}^n$ is defined as

$$\|x\|_* = \max_{y\in\mathbf{R}^n:\|y\|\leq 1} x^T y$$

(a) Prove that $\|\cdot\|_*$ is a valid norm.

(b) Denote $\|x\|_{p,*}$ as the dual norm to the ℓ_p-norm ($p \geq 1$). Show that

$$\|x\|_{p,*} = \|x\|_q, \text{ where } \frac{1}{p} + \frac{1}{q} = 1.$$

Hence, $\|\cdot\|_{2,*} = \|\cdot\|_2$, $\|\cdot\|_{1,*} = \|\cdot\|_\infty$, $\|\cdot\|_{\infty,*} = \|\cdot\|_1$.

Exercise 1.2 (Unit Ball) The *unit ball* of any norm $\|\cdot\|$ is the set

$$B_{\|\cdot\|} = \{x \in \mathbf{R}^n : \|x\| \leq 1\}.$$

One can easily see that $B_{\|\cdot\|}$ is symmetric w.r.t. the origin ($x \in B_{\|\cdot\|}$ if and only $-x \in B_{\|\cdot\|}$) and compact (closed and bounded) with nonempty interior. Show that the set $B_{\|\cdot\|}$ is convex.

Exercise 1.3 (Vice Versa) Let B be convex, symmetric w.r.t. the origin, and compact with nonempty interior. Show that the following function $\|\cdot\|_B$:

$$\|x\|_B = \inf\{t > 0 : \frac{x}{t} \in B\}$$

is a valid norm. Moreover, its unit ball is exactly the set B.

Exercise 1.4 (Neighborhood of Convex Sets) Let $X \subset \mathbf{R}^n$ be a convex set and let $\epsilon > 0$. The *ϵ-neighborhood* of the set X under norm $\|\cdot\|$ is defined as

$$X^\epsilon = \{x \in \mathbf{R}^n : \inf_{y \in X} \|x - y\| \leq \epsilon\}.$$

Prove that X^ϵ is a convex set.

Problem 2: Strong Duality

Recall that in class we have shown the

[Farkas' Lemma] Let $A \in \mathbf{R}^{n \times m}$ and $b \in \mathbf{R}^m$, exactly one of the two sets must be empty:

$$\begin{aligned} (i): &\quad \{x \in \mathbf{R}^n : Ax = b, x \geq 0\} \\ (ii): &\quad \{y \in \mathbf{R}^m : A^T y \leq 0, b^T y > 0\} \end{aligned}$$

Exercise 2.1 (Variant of Farkas' Lemma) Prove the following variant of Farkas' Lemma: exactly one of the following sets must be empty:

$$\begin{aligned} (i): &\quad \{x \in \mathbf{R}^n : Ax \leq b\} \\ (ii): &\quad \{y \in \mathbf{R}^m : y \geq 0, A^T y = 0, b^T y < 0\} \end{aligned}$$

Exercise 2.2 (Duality of Linear Program) The above lemma can be used to derive strong duality of linear programs. Consider the primal and dual pair of linear programs:

$$\min_x \; c^T x \quad \text{s.t. } Ax = b, x \geq 0 \tag{P}$$

$$\max_y \; b^T y \quad \text{s.t. } A^T y \leq c \tag{D}$$

Assume that the problem (P) has an optimal solution x_* with finite optimal value p_*.

(a) Let y be a feasible solution to (D), show that $b^T y \leq p_*$.

(b) Apply the above variant Farkas' lemma to show that the following system

$$\{y : A^T y \leq c, y^T b \geq p_*\}$$

is nonempty, namely, there exists y that satisfies

$$\begin{bmatrix} A^T \\ -b^T \end{bmatrix} y \leq \begin{bmatrix} c \\ -p_* \end{bmatrix}.$$

Problem 3: Convex Functions

A function f is called *log-convex* if $f(x) > 0, \forall x \in \mathbf{dom}(f)$ and $\ln(f(x))$ is convex.

Exercise 3.1 (Basic Properties) Show that

(a) If f is log-convex, then f is also convex.

(b) f is log-convex if and only if $\forall \lambda \in [0, 1], \forall x, y \in \mathbf{dom}(f)$, we have $f(\lambda x + (1 - \lambda)y) \leq f(x)^\lambda f(y)^{1-\lambda}$.

Exercise 3.2 (Examples) Show that the following functions are log-convex.

(a) Exponentials: $f(x) = e^{ax}$ is log-convex on $\mathbf{R}$;

(b) Sum of exponentials: $f(u, v) = e^u + e^v$ is log-convex on $\mathbf{R}^2$.

Exercise 3.3 (Convexity-Preserving Operation) If f and g are log-convex, then $f + g$ is also log-convex.

Problem 4: Convex Sets and Convex Functions

Denote $\mathcal{S}^n$ as the set of real symmetric matrices of size $n \times n$. Denote $\mathcal{S}^n_+$ as the set of positive semi-definite matrices in $\mathcal{S}^n$. Denote $\mathcal{S}^n_{++}$ as the set of positive definite matrices in $\mathcal{S}^n$. The inner product between two matrices in $\mathcal{S}^n$ is defined as $\langle X, Y \rangle = \text{Trace}(X^T Y) = \text{Trace}(XY) = \sum_{k,l=1}^n x_{kl} y_{kl}$.

Exercise 4.1 (Positive Semidefinite Cone) Recall that set C is a *convex cone* if $\forall x, y \in C$, $\lambda_1 x + \lambda_2 y \in C$ for any $\lambda_1, \lambda_2 \geq 0$. Prove that the set $\mathcal{S}^n_+$ is a convex cone.

Exercise 4.2 (Linear Matrix Inequalities) Let $A_1, \ldots, A_k, B \in \mathcal{S}^n$ be symmetric matrices. Show that the following set

$$C = \{x \in \mathbf{R}^k : B - \sum_{i=1}^k x_i A_i \in \mathcal{S}^n_+\}$$

is convex.

Exercise 4.3 (Log-Determinant) Follow the steps below to show that the negative log-determinant function $f(X) = -\ln(\det(X)) : \mathcal{S}^n_{++} \to \mathbf{R}$ is convex. Consequently, $\det(X)^{-1}$ is log-convex.

(a) Define $\phi(t) = -\ln(\det(X + tH))$. Verify that $\phi(t)$ is well defined on an open interval $I(X, H)$ that contains the origin.

(b) Prove that $\phi(t) = \phi(0) - \sum_{i=1}^n \log(1 + t\lambda_i)$, where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of $X^{-1/2} H X^{-1/2}$.

(c) Combine (a) and (b) to show that $f(X)$ is convex on $\mathcal{S}^n_{++}$.

Problem 5: Convex Optimization

Exercise 5.1 (Convex Reformulation) Show that the following optimization problem

$$\begin{aligned} \min_x \quad & f(x) := \max_{k=1,\ldots,n} |\log(a_k^T x) - \log(b_k)| \\ \text{s.t.} \quad & 0 \leq x_i \leq 1, i = 1, \ldots, m \end{aligned}$$

where $a_k \in \mathbf{R}^m, b_k \in \mathbf{R}, k = 1, \ldots, n$ are given, is equivalent to the following optimization problem

$$\begin{aligned} \min_x \quad & \max_{k=1,\ldots,n} h(a_k^T x / b_k) \\ \text{s.t.} \quad & 0 \leq x_i \leq 1, i = 1, \ldots, m \end{aligned}$$

where $h(u) = \max(u, 1/u)$ for $u > 0$. And show that the above problem is a convex program.