Homework #1

Due Feb 15 (Wednesday) at the beginning of class

Please show all work and intermediate steps. Late submission will lead to 0 credit.

Problem 1: Convex Sets

A real-valued function on $\mathbb{R}^n$ is called a norm, denoted as $\| \cdot \|$, if it satisfies the three properties:

- (positivity): $\forall x \in \mathbb{R}^n, \|x\| \geq 0; \|x\| = 0$ if and only if $x = 0$;
- (homogeneity): $\forall x \in \mathbb{R}^n, \alpha \in \mathbb{R}: \|\alpha x\| = |\alpha| \cdot \|x\|;
- (triangle inequality): $\forall x, y \in \mathbb{R}^n: \|x + y\| \leq \|x\| + \|y\|$.

The standard norms on $\mathbb{R}^n$ are the $\ell_p$-norms ($p \geq 1$):

$$\|x\|_p := \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$$

- [Euclidean norm]: $p = 2, \|x\|_2 := \sqrt{\sum_{i=1}^n x_i^2}$;
- [Manhattan norm]: $p = 1, \|x\|_1 := \sum_{i=1}^n |x_i|$;
- [Maximum norm]: $p = \infty, \|x\|_\infty := \max_{1 \leq i \leq n} |x_i|$.

Exercise 1.1 (Dual Norm) The dual norm of $\| \cdot \|$ on $\mathbb{R}^n$ is defined as

$$\|x\|_* = \max_{y \in \mathbb{R}^n: \|y\| \leq 1} x^T y$$

(a) Prove that $\| \cdot \|_*$ is a valid norm.

(b) Denote $\|x\|_{p,*}$ as the dual norm to the $\ell_p$-norm ($p \geq 1$). Show that

$$\|x\|_{p,*} = \|x\|_q, \text{ where } \frac{1}{p} + \frac{1}{q} = 1.$$  

Hence, $\| \cdot \|_{2,*} = \| \cdot \|_2, \| \cdot \|_{1,*} = \| \cdot \|_\infty, \| \cdot \|_{\infty,*} = \| \cdot \|_1$.

Exercise 1.2 (Unit Ball) The unit ball of any norm $\| \cdot \|$ is the set

$$B_{\| \cdot \|} = \{x \in \mathbb{R}^n : \|x\| \leq 1\}.$$

One can easily see that $B_{\| \cdot \|}$ is symmetric w.r.t. the origin ($x \in B_{\| \cdot \|}$ if and only $-x \in B_{\| \cdot \|}$) and compact (closed and bounded) with nonempty interior. Show that the set $B_{\| \cdot \|}$ is convex.

Exercise 1.3 (Vice Versa) Let $B$ be convex, symmetric w.r.t. the origin, and compact with nonempty interior. Show that the following function $\| \cdot \|_B$:

$$\|x\|_B = \inf\{t > 0 : \frac{x}{t} \in B\}$$

is a valid norm. Moreover, its unit ball is exactly the set $B$. 

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Exercise 1.4 (Neighborhood of Convex Sets) Let \( X \subset \mathbb{R}^n \) be a convex set and let \( \epsilon > 0 \). The \( \epsilon \)-neighborhood of the set \( X \) under norm \( \| \cdot \| \) is defined as
\[
X^\epsilon = \{ x \in \mathbb{R}^n : \inf_{y \in X} \| x - y \| \leq \epsilon \}.
\]
Prove that \( X^\epsilon \) is a convex set.

Problem 2: Strong Duality

Recall that in class we have shown the

[Farkas’ Lemma] Let \( A \in \mathbb{R}^{n \times m} \) and \( b \in \mathbb{R}^m \), exactly one of the two sets must be empty:
\[
(i) : \{ x \in \mathbb{R}^n : Ax = b, x \geq 0 \}
\]
\[
(ii) : \{ y \in \mathbb{R}^m : A^T y \leq 0, b^T y > 0 \}
\]

Exercise 2.1 (Variant of Farkas’ Lemma) Prove the following variant of Farkas’ Lemma: exactly one of the following sets must be empty:
\[
(i) : \{ x \in \mathbb{R}^n : Ax \leq b \}
\]
\[
(ii) : \{ y \in \mathbb{R}^m : y \geq 0, A^T y = 0, b^T y < 0 \}
\]

Exercise 2.2 (Duality of Linear Program) The above lemma can be used to derive strong duality of linear programs. Consider the primal and dual pair of linear programs:
\[
\begin{align*}
\min_x & \quad c^T x \\
\text{s.t.} & \quad Ax = b, x \geq 0
\end{align*} \quad (P)
\]
\[
\begin{align*}
\max_y & \quad b^T y \\
\text{s.t.} & \quad A^T y \leq c
\end{align*} \quad (D)
\]
Assume that the problem (\( P \)) has an optimal solution \( x_* \) with finite optimal value \( p_* \).

(a) Let \( y \) be a feasible solution to (\( D \)), show that \( b^T y \leq p_* \).
(b) Apply the above variant Farkas’ lemma to show that the following system
\[
\{ y : A^T y \leq c, y^T b \geq p_* \}
\]
is nonempty, namely, there exists \( y \) that satisfies
\[
\begin{bmatrix} A^T \\ -b^T \end{bmatrix} y \leq \begin{bmatrix} c \\ -p_* \end{bmatrix}.
\]

Problem 3: Convex Functions

A function \( f \) is called log-convex if \( f(x) > 0, \forall x \in \text{dom}(f) \) and \( \ln(f(x)) \) is convex.

Exercise 3.1 (Basic Properties) Show that

(a) If \( f \) is log-convex, then \( f \) is also convex.

(b) \( f \) is log-convex if and only if \( \forall \lambda \in [0, 1], \forall x, y \in \text{dom}(f) \), we have \( f(\lambda x + (1-\lambda)y) \leq f(x)^\lambda f(y)^{1-\lambda} \).

Exercise 3.2 (Examples) Show that the following functions are log-convex.

(a) Exponentials: \( f(x) = e^{ax} \) is log-convex on \( \mathbb{R} \);

(b) Sum of exponentials: \( f(u, v) = e^u + e^v \) is log-convex on \( \mathbb{R}^2 \).
Exercise 3.3 (Convexity-Preserving Operation) If \( f \) and \( g \) are log-convex, then \( f + g \) is also log-convex.

Problem 4: Convex Sets and Convex Functions

Denote \( S^n \) as the set of real symmetric matrices of size \( n \times n \). Denote \( S^n_+ \) as the set of positive semi-definite matrices in \( S^n \). Denote \( S^n_{++} \) as the set of positive definite matrices in \( S^n \). The inner product between two matrices in \( S^n \) is defined as \( \langle X, Y \rangle = \text{Trace}(X^T Y) = \text{Trace}(XY) = \sum_{k,l=1}^{n} x_{kl} y_{kl} \).

Exercise 4.1 (Positive Semidefinite Cone) Recall that set \( C \) is a convex cone if \( \forall x, y \in C, \lambda_1 x + \lambda_2 y \in C \) for any \( \lambda_1, \lambda_2 \geq 0 \). Prove that the set \( S^n_+ \) is a convex cone.

Exercise 4.2 (Linear Matrix Inequalities) Let \( A_1, \ldots, A_k, B \in S^n \) be symmetric matrices. Show that the following set
\[
C = \{ x \in \mathbb{R}^k : B - \sum_{i=1}^k x_i A_i \in S^n_+ \}
\]
is convex.

Exercise 4.3 (Log-Determinant) Follow the steps below to show that the negative log-determinant function \( f(X) = -\ln(\det(X)) : S^n_{++} \rightarrow \mathbb{R} \) is convex. Consequently, \( \det(X)^{-1} \) is log-convex.

(a) Define \( \phi(t) = -\ln(\det(X + tH)) \). Verify that \( \phi(t) \) is well defined on an open interval \( I(X, H) \) that contains the origin.

(b) Prove that \( \phi(t) = \phi(0) - \sum_{i=1}^n \log(1 + t\lambda_i) \), where \( \lambda_1, \ldots, \lambda_n \) are eigenvalues of \( X^{-1/2}HX^{-1/2} \).

(c) Combine (a) and (b) to show that \( f(X) \) is convex on \( S^n_{++} \).

Problem 5: Convex Optimization

Exercise 5.1 (Convex Reformulation) Show that the following optimization problem
\[
\min_x f(x) := \max_{k=1, \ldots, n} |\log(a_k^T x) - \log(b_k)|
\]
s.t. \( 0 \leq x_i \leq 1, i = 1, \ldots, m \)

where \( a_k \in \mathbb{R}^m, b_k \in \mathbb{R}, k = 1, \ldots, n \) are given, is equivalent to the following optimization problem
\[
\min_x \max_{k=1, \ldots, n} h(a_k^T x/b_k)
\]
s.t. \( 0 \leq x_i \leq 1, i = 1, \ldots, m \)

where \( h(u) = \max(u, 1/u) \) for \( u > 0 \). And show that the above problem is a convex program.