IE 521: Convex Optimization	Spring 2017, UIUC

Lecture 7: Convex Functions (part IV) – February 13

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Courtesy warning: These notes do not necessarily cover everything discussed in the class. Please email TA (swang157@illinois.edu) if you find any typos or mistakes.

In this lecture, we cover the following topics

- Closed convex function
- Convex conjugate function
- Conjugate theory
- Introduction to convex programs

References: Bertsekas, Nedich & Ozdaglar, Chapter 7 and Boyd & Vandenbergh Chapter 3.3

7.1 Closed Convex Functions

Recall for a closed convex set X, $\forall x \notin X$, $\exists$ a hyperplane strictly separates x and X, i.e. $\exists (a_x, b_x)$, s.t. $X \subset H_x := \{y : a_x^T y \leq b_x\}$, and $x \notin H_x$. Hence

$$X = \cap_{x \notin X} \{y : A_x^T y \leq b_x\}$$

Any closed convex set is the intersection of closed half-spaces. Intuitively, when translated to convex functions, any convex function with closed and nonempty epigraph is the upper bound of its affine minorants.

Definition 7.1 *A function is closed if its epigraph is a closed set.*

Remark: Any continuous function is closed, but a closed function is not necessarily continuous. For example,

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ +\infty, & o.w. \end{cases}$$

Proposition 7.2 *Let $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$. Then the following are equivalent.*

(i) f is closed;

(ii) every level set is closed;

(iii) f is lower semi-continuous, i.e. $\forall x \in \mathbf{R}^n, \forall \{x_k\} \to x, f(x) \leq \liminf_{k\to\infty} f(x_k)$.

Proof: Self Exercise ■

Remark Note that a convex function needs not to be closed, e.g.

$$f(x) = \begin{cases} 1, & x = 0 \\ 0, & x > 0 \\ +\infty, & o.w \end{cases}$$

Suppose f is a closed convex function, for a given slope $y \in \mathbf{R}^n$, when is an affine function $y^T x - \beta$ an affine minorant of f?

$$\begin{aligned} f(x) \geq y^T x - \beta &\Rightarrow \beta \geq y^T x - f(x), \forall x \\ &\Rightarrow \beta \geq \underbrace{\sup_{x\in\mathbf{R}^n} \left\{y^T x - f(x)\right\}}_{f^*(y)} \end{aligned}$$

The function $f^*(y)$, is known as the conjugate function of f.

7.2 Convex Conjugate Function

Definition 7.3 *Let $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$: The function*

$$f^*(y) = \sup_{x\in\mathbf{R}^n} \left\{y^T x - f(x)\right\} = \sup_{x\in dom(f)} \left\{y^T x - f(x)\right\}$$

is called the conjugate of function f. Also know as Legendre-Fenchel transformation.

Remark:

1. (Fenchel's inequality) $f(x) + f^*(y) \geq x^T y, \forall x, y$
2. f^* is always convex and closed
3. The conjugate of the conjugate function $f^*(y)$, also called *biconjugate*

$$f^{**}(x) = \sup_{y\in\mathbf{R}^n} \left\{x^T y - f^*(y)\right\}$$

   is also closed and convex.

Theorem 7.4 *We have*

(a) $f(x) \geq f^{**}(x)$

(b) If f *is closed and convex, then* $f^{**} = f$*.*

Proof:

(a) By definition of f^*, $f^*(y) \geq y^T x - f(x), \forall y$. This implies that $f(x) \geq y^T x - f^*(y), \forall y$. Hence, $f(x) \geq \sup_y \left\{ y^T x - f^*(y) \right\} = f^{**}(x)$.

(b) Suppose $f^{**} \neq f$, then $\text{epi}(f) \not\subseteq \text{epi}(f^{**})$

$$\exists (x_0, f^{**}(x_0)), \text{ s.t. } (x_0, t_0) \in \text{epi}(f^{**}) \text{ and } (x_0, t_0) \notin \text{epi}(f)$$

Since f is convex and closed, then $\text{epi}(f)$ is a closed and convex set. By separation theorem, $\exists (y, \beta) \neq 0$ s.t. $H = \left\{ (x,t) : y^T x + \beta t = \beta_0 \right\}$ separates $\text{epi}(f)$ and $(x_0, f^{**}(x_0))$, Note $\beta \neq 0$. w.l.o.g. let $\beta = -1$, and

$$y^T x - t > y^T x_0 - f^*(x_0), \forall (x,t) \in \text{epi}(f)$$

This implies that $y^T x_0 - f(x_0) > y^T x_0 - f^{**}(x_0)$. Hence, $f^{**}(x_0) > f(x_0)$. Contradiction!

■

Examples:

1. $f(x) = ax - b$ $\quad f^*(y) = \begin{cases} b, & x = a \\ +\infty, & o.w. \end{cases}$
2. $f(x) = |x|$ $\quad f^*(y) = \begin{cases} 0, & |y| < 1 \\ +\infty, & o.w. \end{cases}$
3. $f(x) = \frac{c}{2} x^2 (c > 0)$ $\quad f^*(y) = \frac{1}{2c} y^2$

7.3 Convex Optimization

The standard form of an optimization is

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \leq 0, i = 1, ..., m \qquad (P) \\ & h_j(x) = 0, j = 1, ..., k \end{aligned}$$

Definition 7.5 *An optimization problem* (P) *is convex if*

1. *the objective function* f *is convex.*

2. *the inequality constraint functions* $g_1, ..., g_m$ *are convex.*
3. *there is either no equality constraint or only linear equality constraint.*

Note the domain of he problem $D = dom(f) \cap dom(g_i)$ *is convex.*

Definition 7.6 *(Feasibility) The set* $C = \{x \in dom(f) : g_i(x) \leq 0, i = 1, ..., m, h_j(x) = 0, j = 1, .., k\}$ *is called the feasible set. Any point* $x \in C$ *is called a feasible solution. We say* (P) *is feasible if* $C \neq \phi$.

Definition 7.7 *(Optimality) The value* $p^* = \inf \{f(x) : g_i(x) \leq 0, i = 1, .., m, h_j(x) = 0, j = 1, ..., k\}$ *is called the optimal value of* (P)*. If* (P) *is infeasible, we set* $p^* = +\infty$.

- *We say* (P) *is unbounded below is* $p^* = -\infty$.
- *We say* (P) *is solvable if* p^* *is finite and exists a feasible solution* x^**, such that* $p^* = f(x^*)$*. We call such* x^**, an optimal solution. The set of all optimal solution is called the optimal set.*

Epigraph form of the problem The standard problem (P) is equivalent as the problem

$$\begin{aligned} \min_{x \in \mathbf{R}^n, t \in \mathbf{R}} \quad & t \\ s.t. \quad & f(x) - t \leq 0 \qquad\qquad (P') \\ & g_i(x) \leq 0, i = 1, ..., m \\ & h_j(x) = 0, j =, 1..., k \end{aligned}$$

Note that

1. (P') is still convex if (P) is convex
2. (x^*, t^*) is optimal to (P') if and only if x^* is optimal to (P) and $t^* = f(x^*)$

Example

$$\begin{aligned} \min_{x} \quad & \max_{j=1,...,m} (a_1^T x + b_1, ..., a_m^T x + b_m) \\ \text{s.t.} \quad & Cx = d \end{aligned}$$

is equivalent to

$$\begin{aligned} \min_{x,t} \quad & t \\ \text{s.t.} \quad & a_i^T x + b_i - t \leq 0, i = 1, ..., m \\ & Cx = d \end{aligned}$$