In this lecture, we cover the following topics

- Path-following Scheme
- Self-concordant Functions

Reference:
Nemirovski, *Interior Point Polynomial Time Methods in Convex Programming*, 2004, Chapter 1
Nesterov, *Introductory Lectures on Convex Optimization*, 2004, Chapter 4.1

17.1 Historical Notes

- 1947: Dantzig, Simplex method for LP
- 1973: Klee and Minty proved that Simplex Method is not a polynomial-time algorithm
- mid-1970s: Shor, Nemirovski and Yudin, Ellipsoid method for linear and convex programs
- 1979: Khachiyan proved the polynomial-time solvability of LP
- 1984: Karmarkar, polynomial algorithm (potential reduction interior point method) for LP
  (1967): Dikin, affine scaling algorithm (simplification of Karmarkar’s algorithm) for LP
- late-1980s: Renegar, Gonzaga, path-following interior point method for LP
- 1988: Nesterov and Nemirovski, extend interior point method for convex programs

17.2 Path following Scheme

We intend to solve a general convex program

\[
\min_x f(x) \\
\text{s.t. } g_i(x) \leq 0, i = 1, ..., m
\]

\[(P)\]

where \( f, g_i \) are twice continuously differentiable convex functions.
Denote $X = \{ x : g_i(x) \leq 0, \forall i = 1, \ldots, m \}$ as the feasible domain. Assume Slater condition holds and $X$ is bounded, so $X$ is a compact convex set with non-empty interior.

**Barrier Method:** solve a series of unconstrained problems

$$\min_x \ t f(x) + F(x) \quad (P_t)$$

where $t > 0$ is a penalty parameter and $F(x)$ is a barrier function that satisfies:

- $F : \text{int}(X) \to \mathbb{R}$ and $F(x) \to +\infty$ as $x \to \partial(X)$
- $F$ is smooth (twice continuously differentiable) and convex
- $F$ is non-degenerate, i.e. $\nabla^2 F(x) \succ 0, \forall x \in \text{int}(X)$

Note that for any $t > 0$, $(P_t)$ has a unique solution in the interior of $X$.

Denote $x^*(t) = \arg\min_x tf(x) + F(x)$

The path $\{ x^*(t), t > 0 \}$ is called the **central path**. We have

$$x^*(t) \to x^*, \text{as } t \to \infty$$

To implement the above path-following scheme, need to specify:

1. the barrier function $F(x)$:
   - use self-concordant barriers, e.g. $F(x) = -\sum_{i=1}^m \log(-g_i(x))$
2. the method to solve unconstrained minimization problems $(P_t)$:
   - use Newton method
3. the policy to update the penalty parameter $t$.

### 17.3 Self-concordant Function

**Definition 17.1** A function $f : \mathbb{R} \to \mathbb{R}$ is self-concordant if $f$ is convex and

$$|f'''(x)| \leq \kappa f''(x)^{3/2}, \forall x \in \text{dom}(f)$$

for some constant $\kappa \geq 0$.

When $\kappa = 2$, $f$ is called a standard self-concordant function.

**Examples:**
• Logarithmic function: \( f(x) = -\ln(x), x > 0 \) is standard self-concordant:

\[
f'(x) = -\frac{1}{x}, f''(x) = \frac{1}{x^2}, f'''(x) = -\frac{2}{x^3}, \frac{|f'''(x)|}{f''(x)^{3/2}} = 2
\]

• Linear function: \( f(x) = cx \) is self-concordant with constant \( \kappa = 0 \):

\[
f'(x) = c, f''(x) = 0, f'''(x) = 0
\]

• Convex quadratic function: \( f(x) = \frac{a}{2}x^2 + bx + c (a > 0) \) is self-concordant with \( \kappa = 0 \):

\[
f'(x) = ax + b, f''(x) = a, f'''(x) = 0
\]

• Exponential function: \( f(x) = e^x \) is not self-concordant:

\[
f'(x) = f''(x) = f'''(x) = e^x, \frac{|f'''(x)|}{f''(x)^{3/2}} = e^{x/2} \to +\infty \text{ as } x \to -\infty
\]

• Power functions:

\[
f(x) = \frac{1}{x^p} (p > 0), (x > 0)
f(x) = |x|^p (p > 2)
f(x) = x^{2p} (p > 2)
\]

are not self-concordant.

**Remark:** Self-concordant function is **affine-invariant**:
If \( f(x) \) is self-concordant, \( \tilde{f}(y) = f(ay+b) \) is also self-concordant with the same constant.

**Proof:** It is easy to see that \( \tilde{f} \) is convex and

\[
\frac{\tilde{f}'''(y)}{\tilde{f}''(y)^{3/2}} = \frac{|a^3 f'''(ay+b)|}{[a^2 f''(ay+b)]^{3/2}} = \frac{f'''(ay+b)}{f''(ay+b)^{3/2}} \leq \kappa
\]

When extending to a function \( f \) defined on \( \mathbb{R}^n \) and \( f \in C^3(\mathbb{R}^n) \), we say \( f \) is self-concordant if it is self-concordant along every line, namely, \( \forall x \in \text{dom}(f), h \in \mathbb{R}^n, \phi(t) = f(x + th) \) is self-concordant with some constant \( \kappa \geq 0 \).

Denote

\[
D^k f(x)[h_1, ..., h_k] = \left. \frac{\partial^k}{\partial t_1 ... \partial t_k} \right|_{t_1 = ... = t_k = 0} f(x + t_1 h_1 + ... + t_k h_k)
\]

as the k-th differential of \( f \) taken at \( x \) along the directions \( h_1, ..., h_k \) e.g.

\[
D f(x)[h] = \phi'(0) = \langle \nabla f(x), h \rangle
\]

\[
D^2 f(x)[h, h] = \phi''(0) = \langle \nabla^2 f(x) h, h \rangle
\]
**Definition 17.2** We say a function $f : \mathbb{R}^n \to \mathbb{R}$ is self-concordant if

$$D^3 f(x)[h, h, h] \leq \kappa (D^2 f(x)[h, h])^{3/2}, \forall x \in \text{dom}(f), h \in \mathbb{R}^n$$

for some constant $\kappa \geq 0$

**Operations Preserving Self-Concordance**

**Proposition 17.3**

1. (Affine invariant) If $f(y)$ is self-concordant with constant $\kappa$, then the function $\tilde{f}(x) = f(Ax + b)$ is also self-concordant with constant $\kappa$.

2. (Summation) If $f_1(x)$ and $f_2(x)$ are self-concordant with constants $\kappa_1$, $\kappa_2$, then the function $\tilde{f}(x) = f_1(x) + f_2(x)$ is self-concordant with constant $\kappa = \max\{\kappa_1, \kappa_2\}$

3. (Scaling) If $f(x)$ is self-concordant with constant $\kappa$, and $\alpha > 0$ then the function $\alpha f(x)$ is also self-concordant with $\kappa = \frac{\kappa}{\sqrt{\alpha}}$

**Proof:** Exercise!

Hence, it is straightforward to see that

**Corollary 17.4** $f(x) = -\sum_{i=1}^{m} \ln(b_i - a_i^T x)$ is standard self-concordant on $\text{int}(X)$, where $X = \{x : a_i^T x \leq b_i, i = 1, ..., m\}$

**Remark:** Indeed, under regular conditions, self-concordant functions are barrier functions:

- If $\text{dom}(f)$ contains no straight line, then $\nabla^2 f(x)$ is non-degenerate (strictly convex)
- If $f$ is closed convex, then $f(x_k) \to +\infty$ if $\{x_k\} \subseteq \text{dom}(f)$ and $x_k \to \bar{x} \in \partial(\text{dom}(f))$