IE 521: Convex Optimization **Spring 2017, UIUC**

Lecture 14: Conic Duality – March 08

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Courtesy warning: These notes do not necessarily cover everything discussed in the class. Please email TA (swang157@illinois.edu) if you find any typos or mistakes.

In this lecture, we cover the following topics

- Dual Cone
- Conic Duality Theorem

References: Ben-Tal & Nemirovski. *Lectures on Modern Convex Optimization*, Chapter 1.4

14.1 Motivation

Recall the LP Duality:

$$(P) \quad \begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned} \qquad\qquad (D) \quad \begin{aligned} \max \quad & b^T y \\ \text{s.t.} \quad & A^T y \leq c \end{aligned}$$

$$(LP) \quad \begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \end{aligned} \qquad\qquad (LD) \quad \begin{aligned} \max \quad & b^T y \\ \text{s.t.} \quad & A^T y = c \\ & y \geq 0 \end{aligned}$$

Now consider the conic program

$$(CP) \quad \begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq_K b \end{aligned} \qquad\qquad (CD) \quad ?$$

Observe that

$$\begin{aligned} Ax \geq b &\Rightarrow \forall y \geq 0, y^T(Ax) \geq y^T b \\ &\Rightarrow \forall y \geq 0 \text{ and } A^T y = c, c^T x \geq b^T y \\ &\Rightarrow c^T x \geq \max\left\{b^T y : y \geq 0, A^T y = c\right\} \end{aligned}$$

Now that $Ax \geq_K b \Rightarrow y^T(Ax) \geq y^T b$ for which y?

14.2 Dual Cone

Definition 14.1 *Let K be a nonempty cone. The set*

$$K_* = \left\{y : y^T x \geq 0, \forall x \in K\right\}$$

is called the dual cone to K. Note that K_* is a closed cone.

Proposition 14.2 *Let K be a closed cone and K_* be its dual cone. Then:*

1. $(K_*)_* = K$
2. *K is pointed iff K_* has non-empty interior*
3. *K is a regular cone iff K_* is a regular cone*

Proof: Self-exercise. ■

Examples: We call these self-dual cones.

- $(\mathbf{R}^m_+)_* = \mathbf{R}^m_+$
- $(L^m)_* = L^m$
- $(S^m_+)_* = S^m_+$

14.3 Conic Duality

The dual of the conic program

$$(CP) \quad \min_x \left\{c^T x : Ax \geq_K b\right\}$$

is given by

$$(CD) \quad \max_y \left\{A^T y = c, y \geq_{K_*} 0\right\}$$

Theorem 14.3 *(Weak Conic Duality) The optimal value of (CD) is a lower bound of the optimal value of (CP).*

Theorem 14.4 *(Strong Conic Duality) If the primal (CP) is bounded below and strictly feasible, $i, e, \exists x_0$, s.t. $Ax_0 \succ_K b$. Then the dual (CD) is solvable and the optimal values equal to each other.*

Proof: Let p^* be the optimal value of the primal (CP). It's sufficient to show that $\exists y^*$ feasible to (CD), such that $b^T y^* \geq p^*$.

Suppose $c = 0$, $p^* = 0$, $\exists y^* = 0$, s.t. $y^* \geq_{K_*}, A^T y^* = c$ and $b^T y^* = p^*$.

Now consider $c \neq 0$. let the set $M = \left\{Ax - b : c^T x \leq p^*\right\}$. Note that M is a nonempty set.

Claim 1 : $M \cap \text{int}(K) = \emptyset$
This is because: Suppose $\exists \bar{x}$, s.t. $A\bar{x} \in \text{int}(K)$ and $c^T \bar{x} \leq p^*$. Then there exists a small enough neighborhood of $\bar{x}$ that are feasible. Since $c \neq 0$, there exists a point $\tilde{x}$ in this neighborhood such that $c^T \tilde{x} < c^T \bar{x} \leq p^*$. This contradicts with the fact that p^* is the optimal value.

By Separation Theorem, $\exists y \neq 0$, s.t.

$$\sup_{z \in M} y^T z \leq \inf_{z \in \text{int}(K)} y^T z$$

Note that $\inf_{z \in \text{int}(K)} y^T z = 0$, since $\text{int}(K)$ contains the ray$\{tz : t \geq 0\}, \forall z \in \text{int}(K)$.

Hence, we have $y \in K^*$ and

$$\sup_{x : c^T x \leq p^*} y^T (Ax - b) \leq 0 \tag{14.1}$$

Therefore, it must hold that $\lambda c = A^T y$ for some $\lambda \geq 0$. otherwise, the supremum goes to infinity.

Claim 2 : $\exists \lambda > 0$, s.t. $A^T y = \lambda c$

This is because: Suppose $\lambda = 0, A^T y = 0$, (14.1) implies $-b^T y \leq 0$. Since (CP) is strictly feasible, $\exists x_0$, s.t. $Ax_0 - b \in \text{int}(K)$. We already show $y \in K_*$, then $y^T (Ax_0 - b) > 0$, which implies $y^T b > 0$. Contradiction!

Now let $y^* = \frac{y}{\lambda}$, we obtain $y^* \in K_*, A^T y^* = c$ and $c^T x - (y^*)^T b \leq 0, \forall x$ such that $c^T x \leq p^*$. Therefore, y^* is dual feasible and $b^T y^* \geq p^*$

■

Corollary 14.5 *If the dual (CD) is bounded above and <u>strictly feasible</u>,*

$$i.e.\ \exists y,\ s.t.\ y >_{K_*} 0 \text{ and } A^T y = c$$

then the primal (CP) is solvable and the optimal values equal to each other.

Corollary 14.6 *If both (CP) and (CD) are strictly feasible, the both are solvable with equal optimal values.*

Theorem 14.7 *(Optimality Conditions) Suppose at least one of (CP) and (CD) is bounded and strictly feasible, then the feasible primal-dual pair (x^*, y^*) is a pair of optimal primal-dual solutions iff*

1. *(Zero duality gap) if and only if $c^T x^* - b^T y^* = 0$*
2. *(Complementary Slackness) if and only if $(Ax^* - b)^T y^* = 0$*

Proof: Let p^* and d^* be the optimal values of (CP) and (CD)

$$c^T x^* - b^T y^* = (c^T x^* - p^*) + (d^* - b^T y^*) + (p^* - d^*)$$

All three terms on RHS are ≥ 0. And $c^T x^* - b^T y^* = 0$ iff $c^T x^* = p^*$, $d^* = b^T y^*$ and $p^* = d^*$ ■

Remark In the case of LP, the strictly feasibility is not required for strong duality and it is not required for a program to be solvable. However, in the genetic case of CP, this is not necessary true.

<u>Example 1</u>: A conic problem can be strictly feasible and bounded, but NOT solvable.

$$\begin{aligned} \min_{x_1,x_2} \quad & x_1 \\ \text{s.t.} \quad & \begin{bmatrix} x_1 - x_2 \\ 1 \\ x_1 + x_2 \end{bmatrix} \geq_{L^3} 0 \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \min_{x_1,x_2} \quad & x_1 \\ \text{s.t.} \quad & 4x_1 x_2 \geq 1 \\ & x_1 + x_2 > 0 \end{aligned}$$

<u>Example 2</u>: A conic problem can be not strictly feasible yet solvable, and the dual is infeasible.

$$\begin{aligned} \min_{x_1,x_2} \quad & x_2 \\ \text{s.t.} \quad & \begin{bmatrix} x_1 \\ x_2 \\ x_1 \end{bmatrix} \geq_{L^3} 0 \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \min_{x_1,x_2} \quad & x_2 \\ \text{s.t.} \quad & x_2 = 0 \\ & x_1 \geq 0 \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \max_{\lambda} \quad & 0 \\ \text{s.t.} \quad & \begin{bmatrix} \lambda_1 + \lambda_3 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ & \lambda \geq_{L^3} 0 \end{aligned}$$