IE 521: Convex Optimization **Spring 2017, UIUC**

Lecture 13: Conic Programming – March 06

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Courtesy warning: These notes do not necessarily cover everything discussed in the class. Please email TA (swang157@illinois.edu) if you find any typos or mistakes.

In this lecture, we cover the following topics

- Generalized Inequality Constraints and Cones
- Conic Programs: LP, SOCP, SDP

References: Ben-Tal & Nemirovski. *Lectures on Modern Convex Optimization*, Chapter 1.4

13.1 Generalized Inequality

In order to extend the linear constraints to nonlinear convex constraints, we may extend the standard component-wise inequality to a generalized vector inequality.

$$Ax \geq b \Rightarrow Ax \succcurlyeq b$$

where the partial ordering "$\succcurlyeq$" also satisfies:

(i) reflexibility: $a \succcurlyeq a$

(ii) anti-symmetry: $a \succcurlyeq b$ and $b \succcurlyeq a$ implies $a = b$

(iii) transitivity: $a \succcurlyeq b$ and $b \succcurlyeq c$ implies $a \succcurlyeq c$

(iv) homogeneity: $a \succcurlyeq b$ and $\lambda \in \mathbf{R}_+$ implies $\lambda a \succcurlyeq \lambda b$

(v) additivity: $a \succcurlyeq b$ and $c \succcurlyeq b$ implies $a + c \succcurlyeq b + d$

Remark 1: The generalized inequality on $\mathbf{R}^m$ is completely identified by the set $K = \{a \in \mathbf{R}^m : a \succcurlyeq 0\}$ via the rule

$$a \succcurlyeq b \Leftrightarrow a - b \in K$$

This is because:

- $a \succcurlyeq b$ and $-b \succcurlyeq -b$ (by (i)) implies $a - b \succcurlyeq 0$
- $a - b \succcurlyeq 0$ and $b \succcurlyeq b$ (by (i)) implies $a \succcurlyeq b$

Remark 2 The set K satisfies

1. K is nonempty: $0 \in K$
2. K is closed w.r.t addition: $a, b \in K \Rightarrow a + b \in K$
3. K is closed w.r.t multiplication $a \in K, \lambda \in \mathbf{R} \Rightarrow \lambda a \in K$
4. K is pointed: $a, -a \in K \Rightarrow a = 0$

Equivalently, K must be a nonempty pointed convex cone. Indeed, this condition is both necessary and sufficient for the set K to define a partial ordering "$\geq_K$" via the rule:

$$a \geq_K b \Leftrightarrow a - b \in K$$

Remark 3: The standard inequality $a \geq b \Leftrightarrow a - b \in \mathbf{R}_+^m$. In addition, $K = \mathbf{R}_+^m$ is also closed and has non-empty interior.

13.2 Conic Programs

Definition 13.1 *Let K be a regular cone (the cone is convex, closed, pointed and with a nonempty interior) and "$\geq_K$" be the induced inequality. The optimization problem:*

$$\begin{aligned} \min_x \quad & c^T x \\ s.t. \quad & Ax \geq_K b \quad (CP) \end{aligned}$$

is called a conic program associated with the cone K.

Examples of Regular Cones

- Non-negative Orthant:
$$\mathbf{R}_+^m = \{x \in \mathbf{R}^m : x_i \geq 0, i = 1, ..., m\}$$
- Lorentz Cone (a.k.a second order/ ice-cream cone)
$$L^m = \left\{ x \in \mathbf{R}^m : x_m \geq \sqrt{\sum_{i=1}^{m-1} x_i^2} \right\}$$
- Semidefine Cone:
$$S_+^m = \{A \in S^m : A \succcurlyeq 0\}$$
is the set of $m \times m$ symmetric positive semidefinite matrices.

Typical Conic Programs

- Linear Program: $K = \mathbf{R}^m_+ = \mathbf{R} \times ... \times \mathbf{R}$ is a direct product of nonnegative orthant:
$$\min_x \left\{ c^T x : a_i^T x - b_i \geq 0, i = 1, ..., m \right\} \tag{LP}$$
where the linear map $Ax - b_i := [a_i^T x - b_i; ....; a_m^T x - b_m]$
- Conic Quadratic Program (a.k.a Second Order Conic Program): $K = L^{m_1} \times ... \times L^{m_k}$
$$\min_x \left\{ c^T x : \| D_i x - d_i \|_2 \leq e_i^T x - f_i, i = 1, ..., m \right\} \tag{SOCP}$$
where the linear map
$$Ax - b := \Big[\underbrace{D_1 x - d_1; e_1^T x - f_1}_{\mathbf{R}^{m_1}}; ...; \underbrace{D_k x - d_k; e_k^T x - f_k}_{\mathbf{R}^{m_k}} \Big]$$
- Semidefinite Program:
$$\min_x \left\{ c^T x : \sum_{i=1}^n x_i A_i - B \succcurlyeq 0 \right\} \tag{SDP}$$
where the linear map $Ax - b = \sum_{i=1}^m x_i A_i - B : \mathbf{R}^n \to S^m$, where $A_1, ..., A_n, B \in S^m$

Remark: $(LP) \subseteq (SOCP)$
This is because $a_i^T x - b_i \geq 0 \Leftrightarrow \begin{bmatrix} 0 \\ a_i^T x - b_i \end{bmatrix} \in L^2$.

Remark: $(SOCP) \subseteq (SDP)$
This is because:
$$\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \in L^m \Leftrightarrow \begin{bmatrix} x_m & x_1 & x_2 & \cdots & x_{m-1} \\ x_1 & x_m & & & \\ x_2 & & x_m & & \\ \vdots & & & \ddots & \\ x_{m-1} & & & & x_m \end{bmatrix} \succcurlyeq 0$$

Lemma 13.2 *(Schur Complement) Let $S = \begin{bmatrix} P & Q^T \\ Q & R \end{bmatrix}$ be symmetric with $R \succ 0$. Then*
$$S \succcurlyeq 0 \text{ if and only if } P - Q^T R^{-1} Q \succcurlyeq 0$$

Proof:
$$\begin{aligned} S \succcurlyeq 0 &\Leftrightarrow \forall u.v : \left[u^T, v^T \right] \begin{bmatrix} P & Q^T \\ Q & R \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \geq 0 \\ &\Leftrightarrow \inf_{u,v} u^T P u + 2 v^T Q u + v^T R v \geq 0 \\ &\Leftrightarrow \inf_u u^T (P - Q^T R^{-1} Q) u \geq 0 \\ &\Leftrightarrow P - Q^T R^{-1} Q \succcurlyeq 0 \end{aligned}$$

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To show $(SOCP) \subseteq (SDP)$

1. If $x \in L^m$ and $x \neq 0$, then $x_m > 0$ and

$$x_m \geq \frac{x_1^2 + ... + x_{m-1}^2}{x_m}$$

   The above lemma implies $Ax \geq 0$. If $x \in L^m$ and $x = 0$, then $Ax = 0 \in S_+^m$.

2. If $Ax \geq 0$ and $Ax \neq 0$, then $x_m > 0$. Then by Schur complement Lemma, we have:

$$x_m - \frac{x_1^2 + ... + x_{m-1}^2}{x_m} \geq 0 \Rightarrow x \in L^m$$

   If $Ax = 0$, then $x = 0 \in L^m$

13.3 Examples

1. L_2-norm minimization
   (a) $\min_{x \in \mathbf{R}^n} \| x \|_2$

   Note that

$$\min_{x \in \mathbf{R}^n} \| x \|_2 \iff \begin{array}{ll} \min\limits_{x,t} & t \\ \text{s.t.} & t \geq \| x \|_2 \end{array} \iff \begin{array}{ll} \min\limits_{x,t} & t \\ \text{s.t.} & \begin{bmatrix} x \\ t \end{bmatrix} \geq_{L^{n+1}} 0 \end{array}$$

   (b) $\min_{x \in \mathbf{R}^n} x^T x$

   Note that

$$\min_{x \in \mathbf{R}^n} x^T x \iff \begin{array}{ll} \min\limits_{x,t} & t \\ \text{s.t.} & t \geq x^T x \end{array} \iff \begin{array}{ll} \min\limits_{x,t} & t \\ \text{s.t.} & \begin{bmatrix} 2x \\ t-1 \\ t+1 \end{bmatrix} \geq_{L^{n+2}} 0 \end{array}$$

2. Quadratic problems: $\min_{x \in \mathbf{R}^n} x^T Q x + q^T x$ where $Q = LL^T \succcurlyeq 0$

   Note that

$$\min_{x \in \mathbf{R}^n} x^T Q x + q^T x \iff \begin{array}{ll} \min\limits_{x,t} & t \\ \text{s.t.} & t \geq x^T Q x + q^T x \end{array} \iff \begin{array}{ll} \min\limits_{x,t} & t \\ \text{s.t.} & \begin{bmatrix} 2L^T x \\ t - q^T x - 1 \\ t - q^T x + 1 \end{bmatrix} \geq_{L^{n+2}} 0 \end{array}$$

3. Spectral norm minimization:
   Spectral norm: $\| A \|_2 = \lambda_{\max}(A)$, where A is symmetric
   Note that

$$\min_{x \in \mathbf{R}^n} \quad \| \sum_{i=1}^{n} x_i A_i \|_2 \quad \Longleftrightarrow \min_{x \in \mathbf{R}^n, t} \quad t \quad \text{s.t.} \quad t \geq \lambda_{\max}(\sum_{i=1}^{n} x_i A_i) \quad \Longleftrightarrow \quad \min_{x,t} \quad t \quad \text{s.t.} \quad tI_m - \sum_{i=1}^{n} x_i A_i \succcurlyeq 0$$

   where $A_1, ..., A_n \in S^m$