Lecture 1: Convex Sets – January 23

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Courtesy warning: These notes do not necessarily cover everything discussed in the class. Please email TA (swang157@illinois.edu) if you find any typos or mistakes.

In this lecture, we cover the following topics

- Topology review
- Convex sets (convex/conic/affine hulls)
- Examples of convex sets
- Calculus of convex sets
- Some nice topological properties of convex sets.

1.1 Topology Review

Let X be a nonempty set in $\mathbf{R}^n$. A point x_0 is called an *interior point* if X contains a small ball around x_0, i.e. $\exists r > 0$, such that $B(x_0, r) := \{x : \|x - x_0\|_2 \leq r\} \subseteq X$. A point x_0 is called a *limit point* if there exits a convergent sequence in X that converges to x_0, i.e. $\exists \{x_n\} \subseteq X$, such that $x_n \to x_0$ as $n \to \infty$.

- The interior of X, denoted as $\text{int}(X)$, is the set of all interior point of X.
- The closure of X, denoted as $\text{cl}(X)$, is the set of all limit points of X.
- The boundary of X, denoted as $\partial(X) = \text{cl}(X)/\text{int}(X)$, is the set of points that belongs to the closure but not in the interior.

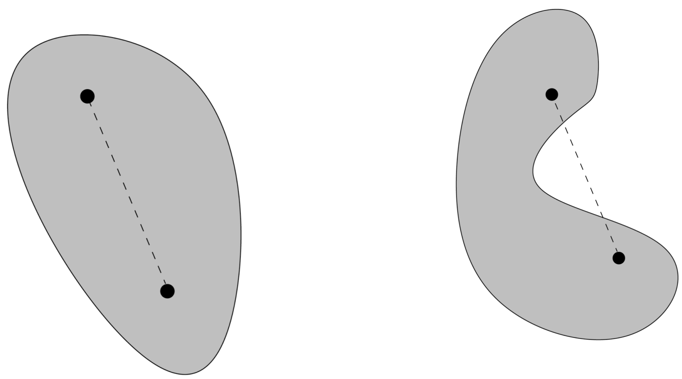
X is *closed* if $\text{cl}(X) = X$; X is *open* if $\text{int}(X) = X$. Here are some basic facts:

- $\text{int}(X) \subseteq X \subseteq \text{cl}(X)$;
- A set X is closed if and only if its complement $X^c = \mathbf{R}^n/X$ is open;
- The intersection of arbitrary number of closed sets is closed, i.e., $\cap_{\alpha \in \mathcal{A}} X_\alpha$ is closed if X_α is closed for all $\alpha \in \mathcal{A}$.
- The union of finite number of closed sets is closed, i.e., $\cup_{i=1}^n X_i$ is closed if X_i is closed for $i = 1, \ldots, n$.

1.2 Convex Sets

Definition 1.1 (Convex set) *A set $X \subseteq \mathbf{R}^n$ is convex if $\forall x, y \in X, \lambda x + (1-\lambda)y \in X$ for any $\lambda \in [0, 1]$.*

In another word, the line segment that connects any two elements lies entirely in the set.



(a) convex set (b) non-convex set

Figure 1.1: Examples of convex and non-convex sets

Given any elements $x_1, \ldots, x_k$, the combination $\lambda_1 x_1 + \lambda_2 x_1 + \ldots + \lambda_k x_k$ is called

- **Covex**: if $\lambda_i \geq 0, i = 1, \ldots, k$ and $\lambda_1 + \lambda_2 + \ldots + \lambda_k = 1$;
- **Conic**: if $\lambda_i \geq 0, i = 1, \ldots, k$;
- **Affine**: if $\lambda_1 + \lambda_2 + \ldots + \lambda_k = 1$;
- **Linear**: if $\lambda_i \in \mathbf{R}, i = 1, \ldots, k$.

Consequently, we have

- A set is *convex* if all convex combinations of its elements are in the set;
- A set is a *convex cone* if all conic combinations of its elements are in the set;
- A set is a *linear subspace* if all affine combinations of its elements are in the set;
- A set is a *linear subspace* if all linear combinations of its elements are in the set.

Clearly, a linear or affine subspace is always a convex cone; a convex cone is always a convex set.

Definition 1.2 (Convex hull) *A convex hull of a set $X \subseteq \mathbf{R}^n$ is the set of all convex combination of its elements, denoted as*

$$Conv(X) = \left\{ \sum_{i=1}^k \lambda_i x_i : k \in \mathbf{N}, \lambda_i \geq 0, \sum_{i=1}^k \lambda_i = 1, x_i \in X, \forall i = 1, \ldots, k \right\}.$$

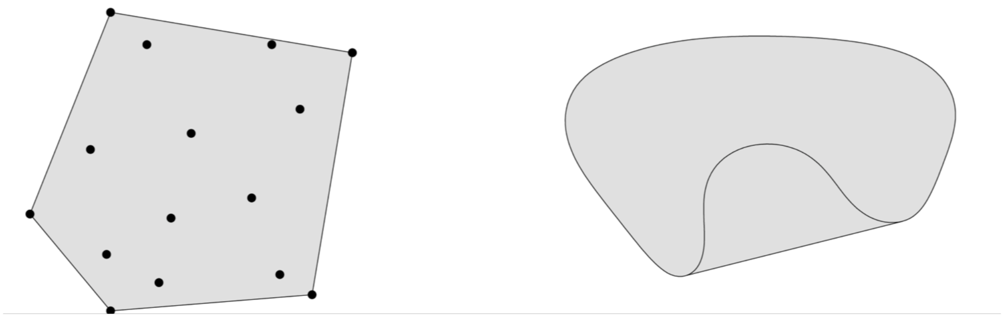


Figure 1.2: Examples of convex hulls

Similarly, one can define the conic hull and affine hull of a set.

$$Cone(X) = \left\{ \sum_{i=1}^k \lambda_i x_i : k \in \mathbf{N}, x_i \in X, \lambda_i \geq 0, \forall i = 1, \ldots, k \right\}.$$

$$Aff(X) = \left\{ \sum_{i=1}^k \lambda_i x_i : k \in \mathbf{N}, x_i \in X, \sum_{i=1}^k \lambda_i = 1, \forall i = 1, \ldots, k \right\}.$$

Proposition 1.3 *We have the following*

1. *A convex hull is always convex.*
2. *If X is convex, then $conv(X) = X$.*
3. *For any set X, $conv(X)$ is the smallest convex set that contains X.*

Proof:

1. By definition, for any $x, y \in \text{Conv}(X)$, we can write $x = \sum_i \lambda_i x_i$ and $y = \sum_i \mu_i x_i$ where $\lambda_i, \mu_i \geq 0$ and $\sum_i \lambda_i = \sum_i \mu_i = 1$. Hence, for any $\alpha \in [0,1]$, we have

$$\alpha x + (1-\alpha) y = \alpha \sum_i \lambda_i x_i + (1-\alpha) \sum_i \mu_i x_i = \sum_i \xi_i x_i$$

where $\xi_i = \alpha \lambda_i + (1-\alpha)\mu_i, \forall i$. Note that $\xi_i \geq 0$ and $\sum_i \xi_i = \alpha \sum_i \lambda_i + (1-\alpha) \sum_i \mu_i = 1$. Therefore, $\alpha x + (1-\alpha) y \in \text{Conv}(X)$. Hence, $\text{Conv}(X)$ is convex.

2. First of all, based on definition of convex hull, it is straightforward to see that $X \subseteq \text{Conv}(X)$. Next, we show that $\text{Conv}(X) \subseteq X$ by induction on k. The baseline is when $k = 1$, which is trivial. Now assuming that any convex combination with k entries is in X, we want to show that any convex combination of $k+1$ entries is still in X. Consider the convex combination below given by $\lambda_1, \ldots, \lambda_{k+1}$ with $\lambda_i \geq 0, i = 1, \ldots, k+1$ and $\sum_{i=1}^{k+1} \lambda_i = 1$.

$$\lambda_1 x_1 + \ldots + \lambda_{k+1} x_{k+1} = (1 - \lambda_{k+1}) \underbrace{\left(\frac{\lambda_1}{1 - \lambda_{k+1}} x_1 + \ldots + \frac{\lambda_k}{1 - \lambda_{k+1}} x_k \right)}_{z} + \lambda_{k+1} x_{k+1}$$

   Based on induction, we can see that $z \in X$ since z is a convex combination of k entries in X. By convexity of X, we further have $\lambda_1 x_1 + \ldots + \lambda_{k+1} x_{k+1} \in X$.

3. Suppose Y is convex and $Y \supseteq X$, we want to show that $Y \supseteq \text{Conv}(X)$. From previous argument, if Y contains X, then Y should contain all convex combinations of X, i.e. $Y \supseteq \text{Conv}(X)$.

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Examples of Convex Sets

Example 1. Some simple convex sets:

- *Hyperplane*: $\{x \in \mathbf{R}^n : a^T x = b\}$
- *Halfspace*: $\{x \in \mathbf{R}^n : a^T x \leq b\}$
- *Affine space*: $\{x \in \mathbf{R}^n : Ax = b\}$
- *Polyhedron*: $\{x \in \mathbf{R}^n : Ax \leq b\}$
- *Simplex*: $\{x \in \mathbf{R}^n : x \geq 0, \sum_{i=1}^n x_i = 1\} = conv(e_1, \ldots, e_n)$.

Example 2. *Euclidean balls*:

$$\{x \in \mathbf{R}^n : \|x\|_2 \leq r\}$$

where $\|\cdot\|_2$ is the Euclidean norm defined on $\mathbf{R}^n$.

Example 3. *Ellipsoid*:

$$\{x \in \mathbf{R}^n : (x-a)^T Q (x-a) \leq r^2\}$$

where $Q \succ 0$ and is symmetric.

1.3 Calculus of Convex Sets

The following operators preserve the convexity of sets, which can be easily verified based on the definition.

1. **Intersection**: If $X_\alpha, \alpha \in \mathcal{A}$ are convex sets, then
$$\cap_{\alpha\in\mathcal{A}} X_\alpha$$
is also a convex set.

2. **Direct product**: If $X_i \subseteq \mathbf{R}^{n_i}, i = 1, \ldots, k$ are convex sets, then
$$X_1 \times \cdots \times X_k := \{(x_1, \ldots, x_k) : x_i \in X_i, i = 1, \ldots, k\}$$
is also a convex set.

3. **Weighted summation**: If $X_i \subseteq \mathbf{R}^n, i = 1, \ldots, k$ are convex sets, then
$$\alpha_1 X_1 + \cdots + \alpha_k X_k := \{\alpha_1 x_1 + \ldots + \alpha_k x_k : x_i \in X_i, i = 1, \ldots, k\}$$
is also a convex set.

4. **Affine image**: If $X \subseteq \mathbf{R}^n$ is a convex set and $\mathcal{A}(x) : x \mapsto Ax + b$ is an affine mapping from $\mathbf{R}^n$ to $\mathbf{R}^k$, then
$$\mathcal{A}(X) := \{Ax + b : x \in X\}$$
is also a convex set.

5. **Inverse affine image**: If $X \subseteq \mathbf{R}^n$ is a convex set and $\mathcal{A}(y) : y \mapsto Ay + b$ is an affine mapping from $\mathbf{R}^k$ to $\mathbf{R}^n$, then
$$\mathcal{A}^{-1}(X) := \{y : Ay + b \in X\}$$
is also a convex set.

Proof:

1. Let $x, y \in \cap_{\alpha\in\mathcal{A}} X_\alpha$, then $x, y \in X_\alpha, \forall\alpha \in \mathcal{A}$. Since X_α is convex, for any $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda)y \in X_\alpha, \forall\alpha \in \mathcal{A}$. Hence, $\lambda x + (1 - \lambda)y \in \cap_{\alpha\in\mathcal{A}} X_\alpha$.

2. Let $x = (x_1, \ldots, x_k) \in X_1 \times \cdots \times X_k, y = (y_1, \ldots, y_k) \in X_1 \times \cdots \times X_k,$. Since X_i is convex, for $\lambda \in [0, 1], \lambda x_i + (1 - \lambda)y_i \in X_i$, for $i = 1, \ldots, k$. Hence
$$\lambda x + (1 - \lambda)y = (\lambda x_1 + (1 - \lambda)y_1, \ldots, \lambda x_k + (1 - \lambda)y_k) \in X_1 \times \cdots \times X_k.$$

3. Let $x, y \in \alpha_1 X_1 + \cdots + \alpha_k X_k$, by definition, there exists $x_i, y_i \in X_i, i = 1, \ldots, k$, such that $x = \alpha_1 x_1 + \ldots + \alpha_k x_k, y = \alpha_1 y_1 + \ldots + \alpha_k y_k$. Hence, for all $\lambda \in [0, 1]$
$$\lambda x + (1 - \lambda)y = \alpha_1 z_1 + \ldots + \alpha_k z_k \in \alpha_1 X_1 + \cdots + \alpha_k X_k$$
because $z_i = \lambda x_i + (1 - \lambda)y_i \in X_i$, for all $i = 1, \ldots, k$.

4. Let $y_1, y_2 \in \mathcal{A}(X)$, then there exits $x_1, x_2 \in X$ such that $y_1 = Ax_1 + b$ and $y_2 = Ax_2 + b$. Therefore, for any $\lambda \in [0,1]$, we have $\lambda y_1 + (1-\lambda)y_2 = A(\lambda x_1 + (1-\lambda)x_2) + b \in \mathcal{A}(X)$ because $\lambda x_1 + (1-\lambda)x_2 \in X$.

5. Let $y_1, y_2 \in \mathcal{A}^{-1}(X)$, then there exits $x_1, x_2 \in X$ such that $x_1 = Ay_1 + b$ and $x_2 = Ay_2 + b$. Therefore, for any $\lambda \in [0,1]$, we have $A(\lambda y_1 + (1-\lambda)y_2) + b \in \lambda x_1 + (1-\lambda)x_2 \in X$, this implies that $\lambda y_1 + (1-\lambda)y_2 \in \mathcal{A}^{-1}(X)$.

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1.4 Nice Topological Properties of Convex Sets

Convex sets are special because of their nice geometric properties.

Proposition 1.4 *If X be a convex set with nonempty interior, then $int(X)$ is dense in $cl(X)$.*

Proof: Let $x_0 \in \text{int}(X)$ and $x \in \text{cl}(X)$. We can construct a convergence sequence $y_n = \frac{1}{n}x_0 + (1 - \frac{1}{n}x$ such that $y_n \to x$. We only need to show that $y_n \in \text{int}(X)$. Therefore, it suffices to prove the following claim:

Claim 1.5 *If $x_0 \in int(X)$ and $x \in cl(X)$, then $[x_0, x) \in int(X)$, namely, for any $\alpha \in [0,1)$, the point $z := \alpha x_0 + (1-\alpha)x \in int(X)$.*

This can be proved as follows. Since $x_0 \in \text{int}(X)$, there exits $r > 0$ such that $B(x_0, r) \subseteq X$. Since $x \in \text{cl}(X)$, there exits a sequence $\{x_n\} \subseteq X$ such that $x_n \to x$. Let $z_n = \alpha x_0 + (1-\alpha)x_n$, then $z_n \to z$. When n is large enough, $\|z_n - z\|_2 \leq \frac{\alpha r}{2}$. Since $B(x_0, r) \subseteq X$ and $x_n \in X$, then $B(z_n, \alpha r) = \alpha B(x_0, r) + (1-\alpha)x_n \subseteq X$. Hence, $B(z, \frac{\alpha r}{2}) \subseteq B(z_n, \alpha r) \subseteq X$. This is because for any $z' \in B(z, \frac{\alpha r}{2})$, $\|z' - z\| \leq \frac{\alpha r}{2}$,

$$\|z' - z_n\|_2 \leq \|z' - z\|_2 + \|z_n - z\|_2 \leq \frac{\alpha r}{2} + \frac{\alpha r}{2} = \alpha r.$$

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Remark. Note that in general, for any set X, $int(X) \subseteq X \subseteq cl(X)$, but $int(X)$ and $cl(X)$ can differ dramatically. For instance, let X be the set of all irrational numbers in $(0,1)$, then $int(X) = \emptyset$, $cl(X) = [0,1]$. The proposition implies that a convex set is perfectly well characterized by its closure or interior if nonempty.