IE 521 Convex Optimization
Homework #2 Solution

your NAME here
your NetID here

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Instructions.

• Homework is due Monday, February 25, at 1:00pm; no late homework accepted.

• Please use the provided \LaTeX\ file as a template.

• You can discuss with others, but please write your own solutions.
Problem 1: Invertible Convex Function

Exercise 1.1 Prove that an invertible real-valued function \( f \) with domain \( \text{dom}(f) \subset \mathbb{R} \) is convex if its inverse function \( f^{-1} \) is convex and monotone decreasing on its domain \( f(\text{dom}(f)) \). (Assume \( \text{dom}(f) \) and \( f(\text{dom}(f)) \) are convex.)

Solution. Function \( f \) is invertible on the domain \( \text{dom}(f) \), so \( \forall x, y \in \text{dom}(f) \), there exist \( u \) and \( v \) such that \( f^{-1}(u) = x \) and \( f^{-1}(v) = y \), and further \( \forall \lambda \in [0, 1] \), we have

\[
 f(\lambda x + (1 - \lambda)y) = f(\lambda f^{-1}(u) + (1 - \lambda)f^{-1}(v)).
\]

By the convexity of \( f^{-1} \), we have

\[
 \lambda f^{-1}(u) + (1 - \lambda)f^{-1}(v) \geq f^{-1}( \lambda u + (1 - \lambda)v).
\]

Since \( f \) is invertible and \( f^{-1} \) is monotone decreasing on its domain \( f(\text{dom}(f)) \), function \( f \) is also monotone decreasing on \( \text{dom}(f) \). Hence, we have

\[
 f(\lambda x + (1 - \lambda)y) = f(\lambda f^{-1}(u) + (1 - \lambda)f^{-1}(v)) \leq f^{-1}(\lambda u + (1 - \lambda)v) = \lambda f(x) + (1 - \lambda)f(y),
\]

which shows that function \( f \) is convex on its convex domain \( \text{dom}(f) \).

Exercise 1.2 The Lambert W function, denoted as \( W(x) \), is the inverse function of \( f(z) = z \exp(z) \). Below is the figure of the real-valued Lambert W function. Note that \( W(x) \) is double-valued on \((-\frac{1}{e}, 0)\) and single-valued on \([0, +\infty)\). We restrict the domain of the Lambert W function to be \([0, +\infty)\) where it is invertible. Prove the Lambert W function is concave on \((0, +\infty)\).

Solution. Similarly, we can prove “an invertible real-valued function \( f \) with a convex domain \( \text{dom}(f) \subset \mathbb{R} \) is concave if its inverse function \( f^{-1} \) is convex and monotone increasing on its convex domain \( f(\text{dom}(f)) \)”.

Consider the inverse function of the Lambert W function on the domain \( \text{dom}(W) = (0, +\infty) \), i.e., function \( f(z) \) on the domain \( W(\text{dom}(W)) = (0, +\infty) \). The first and second derivatives of \( f(z) \) on \((0, +\infty)\) are

\[
 f'(z) = \exp(z) + z \exp(z) > 0;
\]

\[
 f''(z) = 2 \exp(z) + z \exp(z) > 0.
\]

Hence, \( f(z) \) is convex and monotone increasing on the domain \( W(\text{dom}(W)) = (0, +\infty) \), and further the concavity of the Lambert W function on the domain \( \text{dom}(W) = (0, +\infty) \) is proved.
Problem 2: Log-convex and Log-concave Functions

A function $f$ is called log-convex if $f(x) > 0, \forall x \in \text{dom}(f)$ and $\log(f(x))$ is convex. Similarly, a function $f$ is called log-concave if $f(x) > 0, \forall x \in \text{dom}(f)$ and $\log(f(x))$ is concave. For example, these functions $e^{ax}$, $e^{x_1 + e^{x_2}}$ are log-convex.

Exercise 2.1 For each of the following two statements, please decide it’s true or false, and prove it if it’s true or give an counter example if it’s false.

(a) If $f$ is log-convex, then $f$ is also convex.
(b) If $f$ is log-concave, then $f$ is also concave.

Solution. (a) True. Let $g(x) = \log(f(x))$. Since $f$ is log-convex, then $g(x)$ is convex. Note that $f(x) = e^{g(x)}$ is the composition of exponential function (convex and monotonically increasing) and a convex function, then $f$ is also convex.

(b) False. Consider the probability density function of the standard normal distribution $f(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$. The function $\log f(x) = -\frac{1}{2} \log(2\pi) - \frac{x^2}{2}$ which is concave, i.e., $f(x)$ is log-concave. However, the second derivative $f''(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)(x^2-1)$ is not always non-positive, which shows that $f(x)$ is not concave.

Exercise 2.2 Prove that $f$ is log-convex if and only if $\forall \lambda \in [0, 1], \forall x, y \in \text{dom}(f)$, we have

$$f(\lambda x + (1 - \lambda)y) \leq f(x)^{\lambda} f(y)^{1-\lambda}.$$  

Solution. This is because

$$f(x) \text{ is log-convex} \iff \log(f(x)) \text{ is convex}$$

$$\iff \forall x, y \in \text{dom}(f), \forall \lambda \in [0, 1], \log \left( f(\lambda x + (1 - \lambda)y) \right) \leq \lambda \log \left( f(x) \right) + (1 - \lambda) \log \left( f(y) \right)$$

$$\iff \forall x, y \in \text{dom}(f), \forall \lambda \in [0, 1], e^{\log \left( f(\lambda x + (1 - \lambda)y) \right)} \leq e^{\lambda \log \left( f(x) \right) + (1 - \lambda) \log \left( f(y) \right)}$$

$$\iff \forall x, y \in \text{dom}(f), \forall \lambda \in [0, 1], f(\lambda x + (1 - \lambda)y) \leq f(x)^{\lambda} f(y)^{1-\lambda}.$$  

Exercise 2.3 Prove that if $f$ and $g$ are log-convex, then $f + g$ is also log-convex.

Solution. Define a function $h(u, v) = \log(e^u + e^v)$. Function $h(u, v)$ is component-wise non-decreasing in $u$ and $v$. The Hessian of $h(u, v)$ is

$$\nabla^2 h(u, v) = \begin{bmatrix} \frac{e^u e^v}{(e^u + e^v)^2} & -\frac{e^u e^v}{(e^u + e^v)^2} \\ -\frac{e^u e^v}{(e^u + e^v)^2} & \frac{e^u e^v}{(e^u + e^v)^2} \end{bmatrix}.$$  

Since $\nabla^2 h(u, v) \succeq 0$, $h(u, v)$ is jointly convex in $u$ and $v$. Let $u = \log(f)$ and $v = \log(g)$ which are two convex functions. By the preservation of convexity, $h(\log(f) + \log(g)) = \log(f + g)$ is convex and further $f + g$ is log-convex.

Exercise 2.4 Prove that the Lambert W function is log-concave on $(0, \infty)$. 


Solution. Let \( f(x) = \log W(x) \) for \( x \in (0, \infty] \). The second derivative of \( f(x) \) is

\[
 f''(x) = \frac{W''(x)W(x) - [W'(x)]^2}{[W(x)]^2}.
\]

Because \( W(x) \) is positive and strictly concave on \((0, \infty)\), then we have \( W(x) > 0 \) and \( W''(x) < 0 \). Hence, \( f''(x) < 0 \), i.e., the Lambert W function is log-concave on \((0, \infty)\).
Problem 3: Convex Conjugate

Exercise 3.1 (Compute Conjugate) Calculate the conjugate of the following functions:

(a) \( f(x) = e^x \) on \( \mathbb{R} \) (using Lambert W function)
(b) \( f(x) = \frac{1}{2} \|x\|^2 \) on \( \mathbb{R}^n \)
(c) \( f(x) = \log(\sum_{i=1}^n e^{x_i}) \) on \( \mathbb{R}^n \)

Solution. (a) The conjugate for \( f(x) = e^x \) is

\[
f^*(y) = \begin{cases} 
  y \log W(y) - e^{W(y)}, & y > 0 \\
  -1, & y = 0 \\
  +\infty, & \text{o.w.}
\end{cases}
\]

(b) The conjugate for \( f(x) = \frac{1}{2} \|x\|^2 \) is

\[
f^*(y) = \frac{1}{2} \|y\|^2
\]

(c) The conjugate for \( f(x) = \log(\sum_{i=1}^n e^{x_i}) \) is

\[
f^*(y) = \begin{cases} 
  \sum_{i=1}^n y_i \log(y_i), & \text{if } y \geq 0 \text{ and } \sum_{i=1}^n y_i = 1 \\
  +\infty, & \text{o.w.}
\end{cases}
\]

\[\square\]

Exercise 3.2 Let \( f(x) \) and \( g(x) \) be closed convex functions, and \( h(x) = f(x) + g(x) \), then

\[
h^*(y) = \inf_z \{f^*(z) + g^*(y - z)\}
\]

where the latter is the convolution of \( f^* \) and \( g^* \).

[Hint: First show that \((\inf_z \{F(z) + G(y - z)\})^* = F^*(y) + G^*(y)\), and then apply with \( F = f^* \), and \( G = g^* \).]

Solution. First, we prove the following equality \((F \square G)^*(x) = F^*(x) + G^*(x)\) where \(F \square G\) denotes the convolution operator \(F \square G = \inf_y \{F(y) + G(x - y)\}\). This is because

\[
(F \square G)^*(x) = \sup_z \left\{ z^T x - \inf_y \{F(y) + G(z - y)\} \right\}
\]

\[
= \sup_z \left\{ z^T x - \inf_{y_1+y_2=z} \{F(y_1) + G(y_2)\} \right\}
\]

\[
= \sup_z \left\{ \sup_{y_1+y_2=z} \{(y_1 + y_2)^T x - F(y_1) - G(y_2)\} \right\}
\]

\[
= \sup_{y_1,y_2} \{(y_1 + y_2)^T x - F(y_1) - G(y_2)\}
\]

\[
= \sup_{y_1} \{y_1^T x - F(y_1)\} + \sup_{y_2} \{y_2^T x - G(y_2)\}
\]

\[
= F^*(x) + G^*(x)
\]

Using \( F = f^* \) and \( G = g^* \), and the fact that \( F^* = f \) and \( G^* = g \), this leads to

\[
(f^* \square g^*)^*(x) = f(x) + g(x)
\]

Note that \( f + g \) is closed, so taking conjugate on both sides will still hold, i.e. \( f^* \square g^* \) is closed, hence,

\[
(f^* \square g^*)^{**}(x) = (f(x) + g(x))^*
\]
We can also easily show that the convolution of two closed functions is still closed under some condition, i.e. 
\[ f^* \square g^* = (f^* \square g^*)^{**}. \]
Combining these two facts, we arrive at
\[ (f + g)^*(y) = (f^* \square g^*)(y) = \inf \frac{1}{2} \{ f^*(z) + g^*(y - z) \} \]

The condition to make the infimal convolution of two closed proper convex functions \( f^* \) and \( g^* \) is still convex: \( \text{dom}(f^*) \cap \text{dom}(g^*) \neq \emptyset \), every \( d = (d_1, d_2) \) that is a direction of recession of \( f^*, g^* \) and satisfies \( d_1 + d_2 = 0 \), is a direction along which \( f^* + g^* \) is constant. See 3.13 (Infimal Convolution Operation) in Convex Optimization Theory – Chapter 3 – Exercises and Solutions: Extended Version by Dimitri P. Bertsekas.
Problem 4: Revenue Function Is Jointly Concave in Market Shares

There are two products in the market with prices $p_1$ and $p_2$, respectively. The choice probability of product $i = 1, 2$ is given by

$$q_i = \frac{\exp(a_i - b p_i)}{1 + \exp(a_1 - b p_1) + \exp(a_2 - b p_2)},$$

and the probability that a customer doesn’t purchase anything is given by

$$q_0 = \frac{1}{1 + \exp(a_1 - b p_1) + \exp(a_2 - b p_2)}.$$

Assume the parameters $a_1, a_2, b$ are known. This is the so-called Multinomial Logit (MNL) model. Observe that $q_0 + q_1 + q_2 = 1$, so the choice probabilities $q_1$ and $q_2$ can be interpreted as the market shares of products 1 and 2, respectively. Under the MNL model, the expected revenue (price of product 1 * market share of product 1 + price of product 2 * market share of product 2) is

$$R(p_1, p_2) = p_1 q_1 + p_2 q_2 = \frac{p_1 \exp(a_1 - b p_1) + p_2 \exp(a_2 - b p_2)}{1 + \exp(a_1 - b p_1) + \exp(a_2 - b p_2)}.$$

**Exercise 4.1** Rewrite the expected revenue $R$ as a function of market shares $q_1$ and $q_2$ and prove it is jointly concave in market shares $q_1$ and $q_2$.

**Solution.** Since $q_0 + q_1 + q_2 = 1$ and $\frac{q_i}{q_0} = \exp(a_i - b p_i)$ for $i = 1, 2$, we can express $p_i$ as a function of market shares $q_1$ and $q_2$, i.e.,

$$p_i = \frac{a_i}{b} - \frac{1}{b} \log \left( \frac{q_i}{q_0} \right) = \frac{a_i}{b} + \frac{1}{b} \left[ \log(1 - q_1 - q_2) - \log(q_i) \right], \quad i = 1, 2.$$

Therefore, the expected revenue is now a function of market shares $q_1$ and $q_2$

$$R(q_1, q_2) = p_1 q_1 + p_2 q_2 = \frac{a_1 q_1}{b} + \frac{q_1}{b} \left[ \log(1 - q_1 - q_2) - \log(q_1) \right] + \frac{a_2 q_2}{b} + \frac{q_2}{b} \left[ \log(1 - q_1 - q_2) - \log(q_2) \right].$$

The domain of $R(q_1, q_2)$ is defined by $\{(q_1, q_2) : 0 < q_1, q_2 < 1\}$.

Consider a function $\phi(q_1, q_2) = q_1 \cdot \log q_1 - \log(1 - q_2)$ with the domain $\{(q_1, q_2) : 0 < q_1, q_2 < 1\}$. The Hessian of $\phi$ is

$$H_\phi(q_1, q_2) = \begin{bmatrix} \frac{1}{q_1} & \frac{-1}{q_1} \\ \frac{-1}{q_2} & \frac{1}{1-q_2} \end{bmatrix}.$$

For any real numbers $\alpha$ and $\beta$, we have

$$(\alpha \quad \beta) \cdot H_\phi(q_1, q_2) \cdot (\alpha^T \quad \beta^T) = \left( \frac{\alpha}{\sqrt{q_1}} + \frac{\beta \sqrt{q_1}}{1-q_2} \right)^2 \geq 0.$$

Thus, the Hessian of function $\phi(q_1, q_2)$ is positive semidefinite, and function $\phi(q_1, q_2)$ is jointly convex. By the preservation of a convex function, $\phi(q_1, q_1 + q_2)$ is jointly convex, and further $-\phi(q_1, q_1 + q_2)$ is jointly concave. Similarly, we can prove $-\phi(q_2, q_1 + q_2)$ is also jointly concave. Therefore, $R(q_1, q_2)$ is jointly concave in market shares $q_1$ and $q_2$ with the domain $\{(q_1, q_2) : 0 < q_1, q_2 < 1\}$. 

\[\square\]