Instructions.

• Homework is due Monday, February 25, at 1:00pm; no late homework accepted.

• Please use the provided LaTeX file as a template.

• You can discuss with others, but please write your own solutions.
Problem 1: Invertible Convex Function

Exercise 1.1 Prove that an invertible real-valued function $f$ with domain $\text{dom}(f) \subset \mathbb{R}$ is convex if its inverse function $f^{-1}$ is convex and monotone decreasing on its domain $f^{-1}(\text{dom}(f))$.

Solution. Put your solution here.

Exercise 1.2 The Lambert W function, denoted as $W(x)$, is the inverse function of $f(z) = z \exp(z)$. Below is the figure of the real-valued Lambert W function. Note that $W(x)$ is double-valued on $(-\frac{1}{e}, 0)$ and single-valued on $[0, +\infty)$. We restrict the domain of the Lambert W function to be $[0, +\infty)$ where it is invertible. Prove the Lambert W function is concave on $[0, +\infty)$.

Solution. Put your solution here.
Problem 2: Log-convex and Log-concave Functions

A function $f$ is called **log-convex** if $f(x) > 0, \forall x \in \text{dom}(f)$ and $\log(f(x))$ is convex. Similarly, a function $f$ is called **log-concave** if $f(x) > 0, \forall x \in \text{dom}(f)$ and $\log(f(x))$ is concave. For example, these functions $e^{ax}, e^{x_1} + e^{x_2}$ are log-convex.

**Exercise 2.1** For each of the following two statements, please decide it’s true or false, and prove it if it’s true or give an counter example if it’s false.

(a) If $f$ is log-convex, then $f$ is also convex.

(b) If $f$ is log-concave, then $f$ is also concave.

**Solution.** Put your solution here.

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**Exercise 2.2** Prove that $f$ is log-convex if and only if $\forall \lambda \in [0, 1], \forall x, y \in \text{dom}(f)$, we have

$$f(\lambda x + (1 - \lambda)y) \leq f(x)^\lambda f(y)^{1-\lambda}.$$ 

**Solution.** Put your solution here.

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**Exercise 2.3** Prove that if $f$ and $g$ are log-convex, then $f + g$ is also log-convex.

**Solution.** Put your solution here.

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**Exercise 2.4** Prove that the Lambert W function is log-concave on $[0, \infty)$.

**Solution.** Put your solution here.
Problem 3: Convex Conjugate

Exercise 3.1 (Compute Conjugate) Calculate the conjugate of the following functions:

(a) $f(x) = e^{e^x}$ on $\mathbb{R}$ (using Lambert W function)
(b) $f(x) = \frac{1}{2} \|x\|^2$ on $\mathbb{R}^n$
(c) $f(x) = \log(\sum_{i=1}^{n} e^{x_i})$ on $\mathbb{R}^n$

Solution. Put your solution here.

Exercise 3.2 Let $f(x)$ and $g(x)$ be closed convex functions, and $h(x) = f(x) + g(x)$, then

$$h^*(y) = \inf_z \{f^*(z) + g^*(y - z)\}$$

where the latter is the convolution of $f^*$ and $g^*$.

[Hint: First show that $(\inf_z \{F(z) + G(y - z)\})^* = F^*(y) + G^*(y)$, and then apply with $F = f^*$, and $G = g^*$.] 

Solution. Put your solution here.
Problem 4: Revenue Function Is Jointly Concave in Market Shares

There are two products in the market with prices $p_1$ and $p_2$, respectively. The choice probability of product $i = 1, 2$ is given by

$$q_i = \frac{\exp(a_i - bp_i)}{1 + \exp(a_1 - bp_1) + \exp(a_2 - bp_2)},$$

and the probability that a customer doesn’t purchase anything is given by

$$q_0 = \frac{1}{1 + \exp(a_1 - bp_1) + \exp(a_2 - bp_2)}.$$

Assume the parameters $a_1, a_2, b$ are known. This is the so-called Multinomial Logit (MNL) model. Observe that $q_0 + q_1 + q_2 = 1$, so the choice probabilities $q_1$ and $q_2$ can be interpreted as the market shares of products 1 and 2, respectively. Under the MNL model, the expected revenue (price of product 1 * market share of product 1 + price of product 2 * market share of product 2) is

$$R(p_1, p_2) = p_1 q_1 + p_2 q_2 = \frac{p_1 \exp(a_1 - bp_1) + p_2 \exp(a_2 - bp_2)}{1 + \exp(a_1 - bp_1) + \exp(a_2 - bp_2)}.$$

**Exercise 4.1** Rewrite the expected revenue $R$ as a function of market shares $q_1$ and $q_2$ and prove it is jointly concave in market shares $q_1$ and $q_2$.

**Solution.** Put your solution here.