Outline

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Separation Theorems
Definitions of Separation, Strict and Strong Separation
Separation Hyperplane Theorem
Strong Separation Hyperplane Theorem

Theorems of Alternatives
Farkas’ Lemma
LP Duality
Quick Review

- **Radon’s Theorem**
  - Any set of $d + 2$ points in $\mathbb{R}^d$ can be partitioned into two disjoint sets whose convex hulls intersect.

- **Helley’s Theorem**
  - If every $(d + 1)$ of the sets from a collection of $n$ sets in $\mathbb{R}^d$ intersect ($n > d$), then the whole collection of sets intersect.
Question

What’s in common and what’s different?

Figure: Separation of sets
Separation of Sets

**Definition.** Let $S$ and $T$ be two nonempty convex sets in $\mathbb{R}^n$. A hyperplane $H = \{ x \in \mathbb{R}^n : a^T x = b \}$ with $a \neq 0$ is said to separate $S$ and $T$ if $S \cup T \not\subset H$ and

\[
S \subset H^- = \{ x \in \mathbb{R}^n : a^T x \leq b \}
\]
\[
T \subset H^+ = \{ x \in \mathbb{R}^n : a^T x \geq b \}
\]

Figure: Separation of two sets

- Separation is equivalent to say

\[
\sup_{x \in S} a^T x \leq \inf_{x \in T} a^T x \text{ and } \inf_{x \in S} a^T x < \sup_{x \in T} a^T x.
\]
**Strict Separation of Sets**

**Definition.** Let $S$ and $T$ be two nonempty convex sets in $\mathbb{R}^n$. A hyperplane $H = \{x \in \mathbb{R}^n : a^T x = b\}$ with $a \neq 0$ is said to strictly separate $S$ and $T$ if

$$S \subset H^- = \{x \in \mathbb{R}^n : a^T x < b\}$$

$$T \subset H^+ = \{x \in \mathbb{R}^n : a^T x > b\}$$

**Figure:** Strict Separation of two sets
Strong Separation of Sets

**Definition.** Let $S$ and $T$ be two nonempty convex sets in $\mathbb{R}^n$. A hyperplane $H = \{ x \in \mathbb{R}^n : a^T x = b \}$ with $a \neq 0$ is said to **strongly separate** $S$ and $T$ if there exits $b' < b < b''$ such that

$$S \subset \left\{ x \in \mathbb{R}^n : a^T x \leq b' \right\}$$

$$T \subset \left\{ x \in \mathbb{R}^n : a^T x \geq b'' \right\}$$

**Remark.**

- Strong separation $\implies$ strict separation.
- Strict separation $\not\iff$ strong separation.
- Strong separation is equivalent to say

$$\sup_{x \in S} a^T x < \inf_{x \in T} a^T x.$$
Example

In all examples, S and T are two closed and convex sets.

(A) S and T are separated, but not strictly;

(B) S and T are strictly separated, but not strongly;

(C) S and T are strongly separated.
Example

In all examples, $S$ and $T$ are two closed and convex sets

(A) $S$ and $T$ are separated, but not strictly;

(B) $S$ and $T$ are strictly separated, but not strongly;

(C) $S$ and $T$ are strongly separated.
Theorem. Let $S$ and $T$ be two nonempty convex sets. Then $S$ and $T$ can be separated if and only if
\[ \text{rint}(S) \cap \text{rint}(T) = \emptyset. \]
Separation Hyperplane Theorem

**Theorem.** Let $S$ and $T$ be two nonempty convex sets. Then $S$ and $T$ can be separated if and only if

$$\text{rint}(S) \cap \text{rint}(T) = \emptyset.$$  

**Corollary.** Let $S$ be a nonempty convex set and $x_0 \in \partial S$. There exists a supporting hyperplane $H = \{x : a^T x = a^T x_0\}$ with $a \neq 0$ such that

$$S \subset \left\{ x : a^T x \leq a^T x_0 \right\}, \text{ and } x_0 \in H.$$
Proof of Separation Theorem

\[ S \text{ and } T \text{ can be separated iff } \operatorname{rint}(S) \cap \operatorname{rint}(T) = \emptyset. \]

- **Necessity.**
Proof of Separation Theorem

\[ S \text{ and } T \text{ can be separated iff } \text{rint}(S) \cap \text{rint}(T) = \emptyset. \]

- **Necessity.**
  - \( S \text{ and } T \text{ are separated implies that for some } a \neq 0, \)
    \[
    \sup_{x \in S} a^T x \leq \inf_{x \in T} a^T x.
    \]
Proof of Separation Theorem

\[ S \text{ and } T \text{ can be separated iff } \text{rint}(S) \cap \text{rint}(T) = \emptyset. \]

- **Necessity.**
  - \( S \text{ and } T \text{ are separated implies that for some } a \neq 0, \)
    \[ \sup_{x \in S} a^T x \leq \inf_{x \in T} a^T x. \]
  - If \( z \in \text{rint}(S) \cap \text{rint}(T), \) then
    \[ z = \arg\max_{x \in S} \{a^T x\} = \arg\min_{x \in T} \{a^T x\}. \]
Proof of Separation Theorem

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- **Necessity.**
  - $S$ and $T$ are separated implies that for some $a \neq 0$,
    \[ \sup_{x \in S} a^T x \leq \inf_{x \in T} a^T x. \]
  - If $z \in \text{rint}(S) \cap \text{rint}(T)$, then
    \[ z = \arg\max_{x \in S} \{ a^T x \} = \arg\min_{x \in T} \{ a^T x \}. \]
  - The linear function $f(x) = a^T x$ has to be constant on both $S$ and $T$, i.e., $S \cap T \subset H$. (why?)
Proof of Separation Theorem

\[ S \text{ and } T \text{ can be separated iff } \text{rint}(S) \cap \text{rint}(T) = \emptyset. \]

- **Sufficiency.**
Proof of Separation Theorem

\[ S \text{ and } T \text{ can be separated iff } \text{rint}(S) \cap \text{rint}(T) = \emptyset. \]

- **Sufficiency.** Based on constructive steps:
  1. Separation of a convex set \( S \) and \( x_0 \notin \text{cl}(S) \) (key step);
Proof of Separation Theorem

\[ S \text{ and } T \text{ can be separated iff } \text{rint}(S) \cap \text{rint}(T) = \emptyset. \]

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  1. Separation of a convex set \( S \) and \( x_0 \notin \text{cl}(S) \) (**key step**);
  2. Separation of a convex set \( S \) and \( x_0 \notin \text{rint}(S) \);
Proof of Separation Theorem

\[ S \text{ and } T \text{ can be separated iff } \text{rint}(S) \cap \text{rint}(T) = \emptyset. \]

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  1. Separation of a convex set \( S \) and \( x_0 \notin \text{cl}(S) \) (key step);
  2. Separation of a convex set \( S \) and \( x_0 \notin \text{rint}(S) \);
  3. Separation of 0 and \( \text{rint}(S) - \text{rint}(T) \);
Proof of Separation Theorem

$S$ and $T$ can be separated iff $\text{rint}(S) \cap \text{rint}(T) = \emptyset$.

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  1. Separation of a convex set $S$ and $x_0 \notin \text{cl}(S)$ (key step);
  2. Separation of a convex set $S$ and $x_0 \notin \text{rint}(S)$;
  3. Separation of 0 and $\text{rint}(S) - \text{rint}(T)$;
Separation of Convex Set and A Point Outside

**Proposition.** Let $S$ be convex and closed, $x_0 \notin S$. Then $x_0$ and $S$ can be separated.

**Figure:** Separation of a convex set and a point
**Proposition.** Let $S$ be convex and closed, $x_0 \notin S$. Then $x_0$ and $S$ can be separated.

**Proof.** Define

$$d(\{x_0\}, S) := \inf \{ \|x_0 - x\|_2 : x \in S \}$$

$$\text{proj}(x_0) := \arg\min_{x \in S} \{ \|x_0 - x\|_2 \}$$

Figure: Separation of a convex set and a point
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Then $d(\{x_0\}, S) > 0$ and $\text{proj}(x_0)$ exists and is unique (why?). The hyperplane

$$H := \{ x : a^T x = b \}, \quad a = x_0 - \text{proj}(x_0), \quad b = a^T x_0 - \frac{\|a\|_2^2}{2}$$

separates $x_0$ and $S$, i.e. $a^T x < b$, $\forall x \in S$, $a^T x_0 > b$. (why?)

Figure: Separation of a convex set and a point
Strong Separation Hyperplane Theorem

**Theorem.** Let $S$ and $T$ be two nonempty convex sets. Then $S$ and $T$ can be strongly separated if and only if

$$\text{dist}(S, T) := \inf\{\|s - t\|_2 : s \in S, t \in T\} > 0.$$  

In particular, if $S - T$ is closed and $S \cap T = \emptyset$, then $S$ and $T$ can be strongly separated.
Remarks

▶ “$S - T$ is closed” is only a sufficient condition for strong (strict) separation, not a necessary condition.
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- “$S - T$ is closed” is only a sufficient condition for strong (strict) separation, not a necessary condition.

- Even if both $S$ and $T$ are closed convex, $S - T$ might not be closed, and they might not even be strictly separated.
Remarks

- “$S - T$ is closed” is only a sufficient condition for strong (strict) separation, not a necessary condition.

- Even if both $S$ and $T$ are closed convex, $S - T$ might not be closed, and they might not even be strictly separated.

- When both $S$ and $T$ are closed convex, $S \cap T = \emptyset$ and at least one of them is bounded, then $S - T$ is closed, and $S$ and $T$ can be strongly separated.
Proof of Strong Separation Theorem

$S$ and $T$ can be strongly separated iff $\text{dist}(S, T) > 0$.

- **Necessity.**

- **Sufficiency.**
Proof of Strong Separation Theorem

S and T can be strongly separated iff dist(S, T) > 0.

- **Necessity.** If S and T are strongly separated, then
  \[ \exists a \neq 0 : \alpha := \sup_{x \in S} a^T x < \inf_{y \in T} a^T y := \beta. \]

- **Sufficiency.**
Proof of Strong Separation Theorem

\[ S \text{ and } T \text{ can be strongly separated iff } \text{dist}(S, T) > 0. \]

- **Necessity.** If \( S \) and \( T \) are strongly separated, then
  \[ \exists a \neq 0 : \alpha := \sup_{x \in S} a^T x < \inf_{y \in T} a^T y := \beta. \]
  Hence, \( \forall x \in S, y \in T \):
  \[ \|a\|_2 \|y - x\|_2 \geq a^T (y - x) \geq \beta - \alpha \Rightarrow \|y - x\|_2 \geq \frac{\beta - \alpha}{\|a\|_2}. \]

- **Sufficiency.**
Proof of Strong Separation Theorem

S and T can be strongly separated iff dist(S, T) > 0.

► **Necessity.** If S and T are strongly separated, then
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► **Sufficiency.** Suppose \( r := \text{dist}(S, T) > 0, \) then \( (S - T) \)
and \( B(0, r) \) are two disjoint convex sets.
Proof of Strong Separation Theorem

S and T can be strongly separated iff \( \text{dist}(S, T) > 0 \).

- **Necessity.** If S and T are strongly separated, then
  \[ \exists a \neq 0 : \alpha := \sup_{x \in S} a^T x < \inf_{y \in T} a^T y := \beta. \]
  Hence, \( \forall x \in S, y \in T : \)
  \[ \|a\|_2 \|y - x\|_2 \geq a^T (y - x) \geq \beta - \alpha \Rightarrow \|y - x\|_2 \geq \frac{\beta - \alpha}{\|a\|_2}. \]

- **Sufficiency.** Suppose \( r := \text{dist}(S, T) > 0 \), then (\( S - T \)) and \( B(0, r) \) are two disjoint convex sets. By Separation Theorem, \( \exists a \neq 0, \)
  \[ \sup_{z \in S - T} a^T z = \sup_{x \in S, y \in T} a^T (x - y) \leq \inf_{z \in B(0,r)} a^T z < 0. \]
Feasibility of Linear System

Example. Show that the following system have no solution.

\[
\begin{align*}
  x_1 - x_2 + 2x_3 & \leq 0 \\
  -x_1 + x_2 - x_3 & \leq 0 \\
  2x_1 - x_2 + 3x_3 & \leq 0 \\
  4x_1 - x_2 + 10x_3 & > 0
\end{align*}
\]
Feasibility of Linear System

**Example.** Show that the following system have no solution.

\[
\begin{align*}
  x_1 - x_2 + 2x_3 & \leq 0 \quad \cdots \times 3 \\
  -x_1 + x_2 - x_3 & \leq 0 \quad \cdots \times 5 \\
  2x_1 - x_2 + 3x_3 & \leq 0 \quad \cdots \times 3 \\
  4x_1 - x_2 + 10x_3 & > 0
\end{align*}
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Feasibility of Linear System

Example. Show that the following system have no solution.

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&x_1 - x_2 + 2x_3 \leq 0 \quad \cdots \times 3 \\
&-x_1 + x_2 - x_3 \leq 0 \quad \cdots \times 5 \\
&2x_1 - x_2 + 3x_3 \leq 0 \quad \cdots \times 3 \\
&4x_1 - x_2 + 10x_3 > 0
\end{align*}
\]

\[
3 \times Eq.(1) + 5 \times Eq.(2) + 3 \times Eq.(3) \Rightarrow 4x_1 - x_2 + 10x_3 \leq 0.
\]
Feasibility of Linear System

Example. Show that the following system have no solution.

\[
\begin{align*}
\begin{cases}
    x_1 - x_2 + 2x_3 & \leq 0 \quad \cdots \times 3 \\
    -x_1 + x_2 - x_3 & \leq 0 \quad \cdots \times 5 \\
    2x_1 - x_2 + 3x_3 & \leq 0 \quad \cdots \times 3 \\
    4x_1 - x_2 + 10x_3 & > 0
\end{cases}
\end{align*}
\]

\[3 \times \text{Eq.}(1) + 5 \times \text{Eq.}(2) + 3 \times \text{Eq.}(3) \Rightarrow 4x_1 - x_2 + 10x_3 \leq 0.\]

Note the system

\[
\begin{align*}
\begin{cases}
    y_1 - y_2 + 2y_3 & = 4 \\
    -y_1 + y_2 - y_3 & = -1 \\
    2y_1 - y_2 + 3y_2 & = 10 \\
    y_1, y_2, y_3 & \geq 0
\end{cases}
\end{align*}
\]

has a solution \( y = (3, 5, 3). \)
The Celebrated Farkas’ Lemma

**Theorem. (Farkas’ Lemma)** Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. Exactly one of the following sets must be empty:

(i) $\{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$;
(ii) $\{y \in \mathbb{R}^m : A^T y \leq 0, b^T y > 0\}$.

- System (i) and (ii) are often called **strong alternatives**, i.e. exactly one of them must be feasible.
- This is an example of “theorem on alternatives”.

*Figure: Gyula Farkas (1847–1930)*
Geometric View of Farkas’ Lemma

- Let $A = [a_1 | a_2 | \ldots | a_n]$, define the cone
  $\text{Cone} \{a_1, \ldots, a_n\} = \{\sum_{i=1}^{n} x_i a_i : x_i \geq 0, i = 1, \ldots, n\}$

  \[
  \{Ax = b, x \geq 0\} \text{ is infeasible} \iff b \not\in \text{Cone} \{a_1, \ldots, a_n\} \\
  \iff \exists y, y^T a_i \leq 0, \forall i, y^T b > 0
  \]

- Farkas’ lemma can be regarded as a special case of the separation theorem.
Proof of Farkas’ Lemma

- System (i) feasible ⇒ system (ii) infeasible.

- System (i) infeasible ⇒ system (ii) feasible.
Proof of Farkas’ Lemma

- System (i) feasible ⇒ system (ii) infeasible.
  Otherwise, $0 < b^T y = (Ax)^T y = x^T (A^T y) \leq 0$.

- System (i) infeasible ⇒ system (ii) feasible.
Proof of Farkas’ Lemma

- System (i) feasible $\Rightarrow$ system (ii) infeasible. Otherwise, $0 < b^T y = (Ax)^T y = x^T (A^T y) \leq 0$.

- System (i) infeasible $\Rightarrow$ system (ii) feasible. Let $C = \text{Cone} \{a_1, ..., a_n\}$. Then $C$ is convex and closed. And $b \notin C$.
Proof of Farkas’ Lemma

- System (i) feasible $\Rightarrow$ system (ii) infeasible.
  Otherwise, $0 < b^T y = (Ax)^T y = x^T (A^T y) \leq 0$.

- System (i) infeasible $\Rightarrow$ system (ii) feasible.
  Let $C = \text{Cone} \{a_1, \ldots, a_n\}$.
  Then $C$ is convex and closed. And $b \notin C$.

  - By the separation theorem, $b$ and $C$ can be strongly separated, i.e. $\exists y \neq 0 \in \mathbb{R}^m, \gamma \in \mathbb{R}$, s.t.
    $$y^T z \leq \gamma, \forall z \in C, y^T b > \gamma.$$
Proof of Farkas’ Lemma

- System (i) feasible $\Rightarrow$ system (ii) infeasible.
  Otherwise, $0 < b^T y = (Ax)^T y = x^T (A^T y) \leq 0$.

- System (i) infeasible $\Rightarrow$ system (ii) feasible.
  Let $C = \text{Cone} \{ a_1, ..., a_n \}$.
  Then $C$ is convex and closed. And $b \notin C$.
  - By the separation theorem, $b$ and $C$ can be strongly separated, i.e. $\exists y \neq 0 \in \mathbb{R}^m, \gamma \in \mathbb{R}$, s.t.
    \[ y^T z \leq \gamma, \forall z \in C, y^T b > \gamma. \]
  - Since $0 \in C$, we have $\gamma \geq 0$. 

Proof of Farkas’ Lemma
Proof of Farkas’ Lemma

- System (i) feasible ⇒ system (ii) infeasible. Otherwise, \( 0 < b^Ty = (Ax)^Ty = x^T(A^Ty) \leq 0. \)

- System (i) infeasible ⇒ system (ii) feasible.

Let \( C = \text{Cone}\{a_1, \ldots, a_n\}. \)

Then \( C \) is convex and closed. And \( b \not\in C. \)

- By the separation theorem, \( b \) and \( C \) can be strongly separated, i.e. \( \exists y \neq 0 \in \mathbb{R}^m, \gamma \in \mathbb{R}, \) s.t.

\[
y^Tz \leq \gamma, \forall z \in C, y^Tb > \gamma.\]

- Since \( 0 \in C, \) we have \( \gamma \geq 0. \)

- Show that \( \gamma = 0. \) Suppose \( \gamma > 0, \) and \( \exists z_0 \in C \) such that \( y^Tz_0 > 0, \) then \( y^T(\alpha z_0) > \gamma \) for \( \alpha \) large enough.
Proof of Farkas’ Lemma

- System (i) feasible $\Rightarrow$ system (ii) infeasible. Otherwise, $0 < b^T y = (Ax)^T y = x^T (A^T y) \leq 0$.

- System (i) infeasible $\Rightarrow$ system (ii) feasible. Let $C = \text{Cone}\{a_1, \ldots, a_n\}$. Then $C$ is convex and closed. And $b \notin C$.
  
  - By the separation theorem, $b$ and $C$ can be strongly separated, i.e. $\exists y \neq 0 \in \mathbb{R}^m, \gamma \in \mathbb{R}$, s.t.
    \[ y^T z \leq \gamma, \forall z \in C, y^T b > \gamma. \]
  
  - Since $0 \in C$, we have $\gamma \geq 0$.
  - Show that $\gamma = 0$. Suppose $\gamma > 0$, and $\exists z_0 \in C$ such that $y^T z_0 > 0$, then $y^T (\alpha z_0) > \gamma$ for $\alpha$ large enough.
  - Since $a_1, \ldots, a_n \in C$, we have $y^T a_i \leq 0, \forall i = 1, \ldots, m$, i.e., $A^T y \leq 0$. 
Remarks

- The closedness of the cone $\text{Cone}\{a_1, \ldots, a_n\}$ is crucial here. Note that in general, when $S$ is not a finite set, $\text{Cone}(S)$ is not always closed.

- Farkas’ Lemma can also be proved by Fourier-Motzkin elimination.

- Result can be generalized to convex inequalities.
Variant of Farkas’ Lemma

**Theorem.** Exactly one of the following two sets must be empty:

(i) \( \{ x \in \mathbb{R}^n : Ax \leq b \} \)

(ii) \( \{ y \geq 0 : A^T y = 0, b^T y < 0 \} \)

**Theorem.** Exactly one of the following two sets must be empty:

(i) \( \{ x \in \mathbb{R}^n : Ax = b \} \)

(ii) \( \{ y \in \mathbb{R}^m : A^T y = 0, b^T y \neq 0 \} \)
Duality of Linear Program

Consider the primal and dual pair of linear programs

\[
\begin{align*}
\text{(P)} \quad & \min \ c^T x \\
& \text{s.t.} \ Ax = b \\
& \quad \quad x \geq 0 \\
\end{align*}
\]

\[
\begin{align*}
\text{(D)} \quad & \max \ b^T y \\
& \text{s.t.} \ A^T y \leq c
\end{align*}
\]

**Theorem. (LP Duality)** If (P) has a finite optimal value, then so does (D) and the two values equal each other.
Duality of Linear Program

Consider the primal and dual pair of linear programs

\[
\begin{align*}
    \text{min} \quad & c^T x \\
    \text{s.t.} \quad & Ax = b \\
    & x \geq 0 \\
\end{align*}
\]

\[
\begin{align*}
    \text{max} \quad & b^T y \\
    \text{s.t.} \quad & A^T y \leq c \\
\end{align*}
\]

\textbf{Theorem. (LP Duality)} If \((P)\) has a finite optimal value, then so does \((D)\) and the two values equal each other.

\textbf{Proof:} Homework Exercise.
Who introduced LP duality?

**Figure:** Leonid Kantorovich (1912–1986)

**Figure:** George Dantzig (1914–2005)

**Figure:** John von Neumann (1903–1957)
References

- Boyd & Vandenberghe, Chapter 2.5
- Ben-Tal & Nemirovski, Chapter 1.2.5-1.2.6