Outline

Warm-up
  Quick Review
  Questions

Convex Geometry
  Radon’s Theorem
  Helley’s Theorem
  Separation Theorem
Quick Review

- Convex set
  - $X$ is convex iff $\lambda x + (1 - \lambda) y \in X, \forall x, y \in X, \lambda \in [0, 1]$

- Convex hull
  - $\text{Conv}(X) = \left\{ \sum_{i=1}^{k} \lambda_i x_i : k \in \mathbb{N}, \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1, x_i \in X, \forall i \right\}$

- Convexity-preserving operators
  - Taking intersection, Cartesian product, summation
  - Taking affine mapping, inverse affine mapping

- Topological properties
  - For convex sets, $\text{rint}(X)$ is dense in $\text{cl}(X)$

- Representation theorem
  - Any point in the convex hull of set $X$ with dimension $d$ can be written as the convex combination of at most $d + 1$ points in $X$. 
Question 1

Can you find a partition of the sets whose convex hulls intersect?

Figure: Four sets
Question 2

If $I_1$, $I_2$ and $I_3$ are intervals on the real line such that any two have a point in common, do all three have a point in common?
Question 3

Which group is different from others?

Figure: Four groups of disjoint sets
The Mathematicians

Figure: Johann Radon (1887–1956)
Figure: Eduard Helley (1884–1943)
Figure: Hermann Minkowski (1864–1909)
Radon’s Theorem (J. Radon, 1921)

**Theorem.** Let $S$ be a collection of $N$ points in $\mathbb{R}^d$ with $N \geq d + 2$. Then we can write $S = S_1 \cup S_2$ s.t.

$$S_1 \cap S_2 = \emptyset, \text{ and } \text{Conv}(S_1) \cap \text{Conv}(S_2) \neq \emptyset.$$  

**Remark.**

- Any set of $d + 2$ points in $\mathbb{R}^d$ can be partitioned into two disjoint sets whose convex hulls intersect.
- Can be used to show the VC-dimension of the class of halfspaces (linear separators) in $d$-dimensions is $d + 1$.

**Figure:** 3 points separable vs 4 points nonseparable
Proof of Radon’s Theorem

- Let $S = \{x_1, \ldots, x_N\}$ with $N \geq d + 2$.
- Consider the linear system

\[
\begin{align*}
\sum_{i=1}^{N} \gamma_i x_i &= 0 \\
\sum_{i=1}^{N} \gamma_i &= 0
\end{align*}
\]

$(d + 1)$ equations

$N \geq (d + 2)$ unknowns

So there exists a non-zero solution $\gamma_1, \ldots, \gamma_N$.

- Let $I = \{i : \gamma_i \geq 0\}$, $J = \{j : \gamma_j < 0\}$ and $a = \sum_{i \in I} \gamma_i = -\sum_{j \in J} \gamma_j$, then

\[
\sum_{i \in I} \gamma_i x_i = \sum_{j \in J} (-\gamma_j) x_j \Rightarrow \sum_{i \in I} \frac{\gamma_i}{a} x_i = \sum_{j \in J} \frac{-\gamma_j}{a} x_j
\]

- The partition $S_1 = \{x_i, i \in I\}$ and $S_2 = \{x_j : j \in J\}$ gives the desired result.
If $I_1$, $I_2$ and $I_3$ are intervals on the real line such that any two have a point in common, do all three have a point in common?
Helley’s Theorem (E. Helly, 1923)

**Theorem.** Let $S_1, \ldots, S_N$ be a collection of convex sets in $\mathbb{R}^d$ with $N > d$. Assume every $(d + 1)$ sets of them have a point in common, then all the sets have a point in common.

**Figure:** Four convex sets in $\mathbb{R}^2$

Q. Does the theorem still hold if we relax $N = \infty$?
Q. Does the theorem still hold if we relax $(d + 1)$ sets to $d$ sets?
Helley’s Theorem (E. Helly, 1923)

**Theorem.** Let \( S_1, \ldots, S_N \) be a collection of convex sets in \( \mathbb{R}^d \) with \( N > d \). Assume every \((d + 1)\) sets of them have a point in common, then all the sets have a point in common.

**Remark.**

- Not true for infinite collection:
  - E.g. \( S_i = [i, \infty), \cap_{i=1}^{+\infty} S_i = \emptyset \)
- Not true if reduce \((d + 1)\) sets to \(d\) sets.

**Corollary.** Let \( \{S_\alpha\} \) be any collection of *compact* convex sets in \( \mathbb{R}^d \). If every \((d + 1)\) sets have a point in common, then all sets have points in common.
Proof of Helley’s Theorem

Figure: Illustration of $N = 4$, $d = 2$
Proof of Helley’s Theorem

By induction on $N$.

- **Base case:** $N = d + 1$, obviously true.

- **Induction step:** Assume the collection of $N(\geq d + 1)$ sets have common point if every $(d + 1)$ of them have common point. Show this holds for $N + 1$ sets.

- From the assumption, $\exists \{x_1, x_2, \ldots, x_{N+1}\}$ such that $x_i \in S_1 \cap \ldots \cap S_{i-1} \cap S_{i+1} \cap \ldots \cap S_{N+1} \neq \emptyset$.

- By Radon’s theorem, we can split it into two disjoint sets, $\{x_1, \ldots, x_k\}$ and $\{x_{k+1}, \ldots, x_N\}$, and

  $$\text{Conv}(\{x_1, \ldots, x_k\}) \cap \text{Conv}(\{x_{k+1}, \ldots, x_{N+1}\}) \neq \emptyset.$$ 

- Let $z \in \text{Conv}(\{x_1, \ldots, x_k\}) \cap \text{Conv}(\{x_{k+1}, \ldots, x_{N+1}\})$. It can be shown that $z \in S_1 \cap \ldots \cap S_{N+1}$ (why?).
Application of Helley’s Theorem

**Baby Theorem** Let $X$ contain a finite set of points in the plane, such that every three of them are contained in a disk of radius 1. Then $X$ is contained in a disk of radius 1.

**Jung’s Theorem.** Let $X$ contain a finite set of points in the plane, such that any two of them has distance no greater than 1. Then $X$ is contained in a disk of radius $1/\sqrt{3}$.

**Jung’s Theorem.** Let $X \subset \mathbb{R}^n$ be a compact set such that any two of them has Euclidean distance no greater than 1. Then $X$ is contained in a ball with radius $\sqrt{\frac{n}{2(n+1)}}$. 
Application of Helley’s Theorem

Question. Consider the optimization problem

$$p_\star = \min_{x \in \mathbb{R}^{10}} g_0(x), \quad \text{s.t. } g_i(x) \leq 0, \ i = 1, \ldots, 521.$$  

- Suppose $\forall t \in \mathbb{R}, \ X_0 = \{x \in \mathbb{R}^{10} : g_0(x) \leq t\}$ is convex, $X_i = \{x \in \mathbb{R}^{10} : g_i(x) \leq 0\}$ is convex.

- How many constraints can you drop without affecting the optimal value?
Other Applications of Helley’s Theorem

Helley’s theorem is a very fundamental result in convex geometry and can be applied to show many results.

- The centerpoint theorem
- Farkas Lemma
- Sion-Kakutani Theorem
- Chebyshev approximation
When can we separate two sets?

Figure: Four groups of disjoint sets
**Definition.** Let $S$ and $T$ be two nonempty convex sets in $\mathbb{R}^n$. A hyperplane $H = \{x \in \mathbb{R}^n : a^T x = b\}$ with $a \neq 0$ is said to separate $S$ and $T$ if $S \cup T \not\subset H$ and

\[
S \subset H^- = \left\{ x \in \mathbb{R}^n : a^T x \leq b \right\}
\]

\[
T \subset H^+ = \left\{ x \in \mathbb{R}^n : a^T x \geq b \right\}
\]

**Figure:** Separation of two sets
**Strict Separation of Sets**

**Definition.** Let $S$ and $T$ be two nonempty convex sets in $\mathbb{R}^n$. A hyperplane $H = \{x \in \mathbb{R}^n : a^T x = b\}$ with $a \neq 0$ is said to strictly separate $S$ and $T$ if

\[
S \subset H^{-} = \{x \in \mathbb{R}^n : a^T x < b\}
\]

\[
T \subset H^{+} = \{x \in \mathbb{R}^n : a^T x > b\}
\]

*Figure: Strict Separation of two sets*
Strong Separation of Sets

**Definition.** Let $S$ and $T$ be two nonempty convex sets in $\mathbb{R}^n$. A hyperplane $H = \{x \in \mathbb{R}^n : a^T x = b\}$ with $a \neq 0$ is said to strongly separate $S$ and $T$ if there exits $b' < b < b''$ such that

$$S \subset \{x \in \mathbb{R}^n : a^T x \leq b'\}$$

$$T \subset \{x \in \mathbb{R}^n : a^T x \leq b''\}$$

**Remark.**

- Strict separation does not necessarily imply strong separation.
- Strong separation is equivalent to say

$$\sup_{x \in S} a^T x < \inf_{x \in T} a^T x.$$
Separation Hyperplane Theorem

**Theorem.** Let $S$ and $T$ be two nonempty convex sets. Then $S$ and $T$ can be separated if and only if

$$\text{rint}(S) \cap \text{rint}(T) = \emptyset.$$


**Supporting Hyperplane Theorem**

**Theorem.** Let $S$ be a nonempty convex set and $x_0 \in \partial S$. Then there exists a hyperplane $H = \{x : a^T x = a^T x_0\}$ with $a \neq 0$ such that

$$S \subset \{x : a^T x \leq a^T x_0\}, \text{ and } x_0 \in H.$$

![Supporting hyperplane](image)

**Figure:** Supporting hyperplane

- This follows directly from the previous theorem.
- Such a hyperplane is called a supporting hyperplane.
**Theorem.** Let $S$ be closed and convex and $x_0 \notin S$, then there exists a hyperplane that strictly separates $x_0$ and $S$.

**Figure:** Strict separation

- Closedness of the set is crucial here.
- Separating hyperplane can be constructed based on the projection.
Strict Separation Hyperplane Theorem II

**Theorem.** Let \( S \) and \( T \) be two nonempty convex sets and \( S \cap T = \emptyset \). If \( S - T \) is closed, then \( S \) and \( T \) can be strictly separated.

**Remark.**

- Even if both \( S \) and \( T \) are closed convex, \( S - T \) might not be closed, and they might not be strictly separated.

- When both \( S \) and \( T \) are closed convex, \( S \cap T = \emptyset \) and at least one of them is bounded, then \( S - T \) is closed, and \( S \) and \( T \) can be strictly separated.
References

- Boyd & Vandenberghe, Chapter 2.5
- Ben-Tal & Nemirovski, Chapter 1.2.2-1.2.6