CVX Tutorial

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(Slides Origin: Boyd & Vandenberghe)
Cone program solvers

- **LP solvers**
  - many, open source and commercial

- **cone solvers**
  - each handles combinations of a subset of LP, SOCP, SDP, EXP cones
  - open source: SDPT3, SeDuMi, CVXOPT, CSDP, ECOS, SCS, ...
  - commercial: Mosek, Gurobi, Cplex, ...
Transforming problems to cone form

- lots of tricks for transforming a problem into an equivalent cone program
  - introducing slack variables
  - introducing new variables that upper bound expressions

- these tricks greatly extend the applicability of cone solvers

- writing code to carry out this transformation is painful

- **modeling systems** automate this step
Modeling systems

a typical modeling system

- automates transformation to cone form; supports
  - declaring optimization variables
  - describing the objective function
  - describing the constraints
  - choosing (and configuring) the solver

- when given a problem instance, calls the solver

- interprets and returns the solver’s status (optimal, infeasible, . . . )

- (when solved) transforms the solution back to original form
Some current modeling systems

- AMPL & GAMS (proprietary)
  - developed in the 1980s, still widely used in traditional OR
  - no support for convex optimization
- YALMIP (‘Yet Another LMI Parser’, matlab)
  - first object-oriented convex optimization modeling system
- CVX (matlab)
- CVXPY (python, GPL)
- Convex.jl (Julia, GPL, merging into JUMP)
- CVX, CVXPY, and Convex.jl collectively referred to as CVX*
Disciplined convex programming

- describe objective and constraints using expressions formed from
  - a set of basic atoms (affine, convex, concave functions)
  - a restricted set of operations or rules (that preserve convexity)

- modeling system keeps track of affine, convex, concave expressions

- rules ensure that
  - expressions recognized as convex are convex
  - but, some convex expressions are not recognized as convex

- problems described using DCP are convex by construction

- all convex optimization modeling systems use DCP
CVX

- uses DCP
- runs in Matlab, between the `cvx_begin` and `cvx_end` commands
- relies on SDPT3 or SeDuMi (LP/SOCP/SDP) solvers
- refer to user guide, online help for more info
- the CVX example library has more than a hundred examples
Example: Constrained norm minimization

\[
\begin{align*}
A &= \text{randn}(5, 3); \\
b &= \text{randn}(5, 1); \\
\text{cvx_begin} \\
\quad \text{variable } x(3); \\
\quad \text{minimize}(\text{norm}(A*x - b, 1)) \\
\quad \text{subject to} \\
\quad &\quad -0.5 \leq x; \\
\quad &\quad x \leq 0.3; \\
\text{cvx_end}
\end{align*}
\]

\[
\begin{align*}
\min_{x} & \quad \|Ax - b\|_1 \\
\text{s.t.} & \quad -0.5 \leq x \leq 0.3
\end{align*}
\]

- between `cvx_begin` and `cvx_end`, \( x \) is a CVX variable
- statement `subject to` does nothing, but can be added for readability
- inequalities are interpreted elementwise
What CVX does

after cvx_end, CVX

• transforms problem into an LP
• calls solver SDPT3
• overwrites (object) x with (numeric) optimal value
• assigns problem optimal value to cvx_optval
• assigns problem status (which here is Solved) to cvx_status

(had problem been infeasible, cvx_status would be Infeasible and x would be NaN)
Declare variables

- declare variables with variable name[(dims)] [attributes]
  - variable x(3);
  - variable C(4,3);
  - variable S(3,3) symmetric;
  - variable D(3,3) diagonal;
  - variables y z;
Affine expressions

- form affine expressions

\[ A = \text{randn}(4, 3); \]
\[ \text{variables } x(3) \ y(4); \]
- $3x + 4$
- $Ax - y$
- $x(2:3)$
- $\text{sum}(x)$
## Some functions

<table>
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<tr>
<th>function</th>
<th>meaning</th>
<th>attributes</th>
</tr>
</thead>
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<tr>
<td>norm(x, p)</td>
<td>$|x|_p$</td>
<td>cvx</td>
</tr>
<tr>
<td>square(x)</td>
<td>$x^2$</td>
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<tr>
<td>square_pos(x)</td>
<td>$(x_+)^2$</td>
<td>cvx, nondecr</td>
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</tr>
<tr>
<td>max(x)</td>
<td>$\max{x_1, \ldots, x_n}$</td>
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</tr>
<tr>
<td>quad_over_lin(x,y)</td>
<td>$x^2/y$ ($y &gt; 0$)</td>
<td>cvx, nonincr in $y$</td>
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<tr>
<td>lambda_max(X)</td>
<td>$\lambda_{\max}(X)$ ($X = X^T$)</td>
<td>cvx</td>
</tr>
<tr>
<td>huber(x)</td>
<td>$\begin{cases} x^2, &amp;</td>
<td>x</td>
</tr>
</tbody>
</table>
Composition rules

- can combine atoms using valid composition rules, e.g.:
  - a convex function of an affine function is convex
  - the negative of a convex function is concave
  - a convex, nondecreasing function of a convex function is convex
  - a concave, nondecreasing function of a concave function is concave

- for convex $h$, $h(g_1, \ldots, g_k)$ is recognized as convex if, for each $i$,
  - $g_i$ is affine, or
  - $g_i$ is convex and $h$ is nondecreasing in its $i$th arg, or
  - $g_i$ is concave and $h$ is nonincreasing in its $i$th arg

- for concave $h$, $h(g_1, \ldots, g_k)$ is recognized as concave if, for each $i$,
  - $g_i$ is affine, or
  - $g_i$ is convex and $h$ is nonincreasing in $i$th arg, or
  - $g_i$ is concave and $h$ is nondecreasing in $i$th arg
Valid (recognized) examples

u, v, x, y are scalar variables; X is a symmetric $3 \times 3$ variable

• convex:
  
  – $\text{norm}(A\cdot x - y) + 0.1\cdot\text{norm}(x, 1)$
  – $\text{quad\_over\_lin}(u - v, 1 - \text{square}(v))$
  – $\text{lambda\_max}(2\cdot X - 4\cdot\text{eye}(3))$
  – $\text{norm}(2\cdot X - 3, \text{’fro’})$

• concave:
  
  – $\text{min}(1 + 2\cdot u, 1 - \text{max}(2, v))$
  – $\text{sqrt}(v) - 4.55\cdot\text{inv\_pos}(u - v)$
Rejected examples

$u, v, x, y$ are scalar variables

- neither convex nor concave:
  - $\text{square}(x) - \text{square}(y)$
  - $\text{norm}(A \cdot x - y) - 0.1 \cdot \text{norm}(x, 1)$

- rejected due to limited DCP ruleset:
  - $\sqrt{\text{sum}((\text{square}(x)))}$ (is convex; could use $\text{norm}(x)$)
  - $\text{square}(1 + x^2)$ (is convex; could use $\text{square}_\text{pos}(1 + x^2)$, or $1 + 2 \cdot \text{pow}_\text{pos}(x, 2) + \text{pow}_\text{pos}(x, 4)$)
Sets

• some constraints are more naturally expressed with convex sets

• sets in CVX work by creating unnamed variables constrained to the set

• examples:
  - semidefinite(n)
  - nonnegative(n)
  - simplex(n)
  - lorentz(n)

• semidefinite(n), say, returns an unnamed (symmetric matrix) variable that is constrained to be positive semidefinite
Using the semidefinite cone

variables: $X$ (symmetric matrix), $z$ (vector), $t$ (scalar)
constants: $A$ and $B$ (matrices)

- $X == \text{semidefinite}(n)$
  - means $X \in S^n_+$ (or $X \succeq 0$)

- $A*X*A' - X == B*\text{semidefinite}(n)*B'$
  - means $\exists Z \succeq 0$ so that $AXA^T - X = BZB^T$

- $[X \; z; \; z' \; t] == \text{semidefinite}(n+1)$
  - means $\begin{bmatrix} X & z \\ z^T & t \end{bmatrix} \succeq 0$
Objectives and constraints

- **Objective** can be
  - minimize(convex expression)
  - maximize(concave expression)
  - omitted (feasibility problem)

- **Constraints** can be
  - convex expression <= concave expression
  - concave expression >= convex expression
  - affine expression == affine expression
  - omitted (unconstrained problem)
More involved example

A = randn(5);
A = A'*A;
cvx_begin
    variable X(5, 5) symmetric;
    variable y;
    minimize(norm(X) - 10*sqrt(y))
    subject to
        X - A == semidefinite(5);
        X(2,5) == 2*y;
        X(3,1) >= 0.8;
        y <= 4;

cvx_end
Defining new functions

- can make a new function using existing atoms

**example:** the convex deadzone function

\[
f(x) = \max\{|x| - 1, 0\} = \begin{cases} 
0, & |x| \leq 1 \\
x - 1, & x > 1 \\
1 - x, & x < -1 
\end{cases}
\]

- create a file `deadzone.m` with the code

```matlab
function y = deadzone(x)
y = max(abs(x) - 1, 0)
```

- `deadzone` makes sense both within and outside of CVX
Defining functions via incompletely specified problems

• suppose \( f_0, \ldots, f_m \) are convex in \((x, z)\)

• let \( \phi(x) \) be optimal value of convex problem, with variable \( z \) and parameter \( x \)

\[
\begin{align*}
\text{minimize} & \quad f_0(x, z) \\
\text{subject to} & \quad f_i(x, z) \leq 0, \quad i = 1, \ldots, m \\
& \quad A_1x + A_2z = b
\end{align*}
\]

• \( \phi \) is a convex function

• problem above sometimes called *incompletely specified* since \( x \) isn’t (yet) given

• an incompletely specified concave maximization problem defines a concave function
CVX functions via incompletely specified problems

Implement in cvx with

function cvx_optval = phi(x)
cvx_begin
    variable z;
    minimize(f0(x, z))
    subject to
        f1(x, z) <= 0; ...
        A1*x + A2*z == b;
cvx_end

function phi will work for numeric x (by solving the problem)

function phi can also be used inside a CVX specification, wherever a convex function can be used
Simple example: Two element max

• create file max2.m containing

```matlab
function cvx_optval = max2(x, y)
cvx_begin
    variable t;
    minimize(t)
    subject to
        x <= t;
        y <= t;
cvx_end
```

• the constraints define the epigraph of the max function

• could add logic to return \( \max(x, y) \) when \( x, y \) are numeric
  (otherwise, an LP is solved to evaluate the max of two numbers!)
A more complex example

• $f(x) = x + x^{1.5} + x^{2.5}$, with $\text{dom } f = \mathbb{R}_+$, is a convex, monotone increasing function

• its inverse $g = f^{-1}$ is concave, monotone increasing, with $\text{dom } g = \mathbb{R}_+$

• there is no closed form expression for $g$

• $g(y)$ is optimal value of problem

$$\begin{align*}
\text{maximize} & \quad t \\
\text{subject to} & \quad t_+ + t_+^{1.5} + t_+^{2.5} \leq y
\end{align*}$$

(for $y < 0$, this problem is infeasible, so optimal value is $-\infty$)
• implement as
  function cvx_optval = g(y)
  cvx_begin
      variable t;
      maximize(t)
      subject to
          pos(t) + pow_pos(t, 1.5) + pow_pos(t, 2.5) <= y;
  cvx_end

• use it as an ordinary function, as in $g(14.3)$, or within CVX as a concave function:
  cvx_begin
      variables x y;
      minimize(quad_over_lin(x, y) + 4*x + 5*y)
      subject to
          g(x) + 2*g(y) >= 2;
  cvx_end
Example

• optimal value of LP

\[ f(c) = \inf \{ c^T x \mid Ax \preceq b \} \]

is concave function of \( c \)

• by duality (assuming feasibility of \( Ax \preceq b \)) we have

\[ f(c) = \sup \{ -\lambda^T b \mid A^T \lambda + c = 0, \lambda \succeq 0 \} \]
• define $f$ in CVX as

```matlab
function cvx_optval = lp_opt_val(A,b,c)
cvx_begin
    variable lambda(length(b));
    maximize(-lambda'*b);
    subject to
        A'*lambda + c == 0; lambda >= 0;
    cvx_end
```

• in `lp_opt_val(A,b,c)` A, b must be constant; c can be affine
CVX hints/warnings

- watch out for = (assignment) versus == (equality constraint)
- $X \geq 0$, with matrix $X$, is an elementwise inequality
- $X \geq \text{semidefinite}(n)$ means: $X$ is elementwise larger than some positive semidefinite matrix (which is likely not what you want)
- writing subject to is unnecessary (but can look nicer)
- many problems traditionally stated using convex quadratic forms can posed as norm problems (which can have better numerical properties): $x'Px \leq 1$ can be replaced with $\text{norm}(\text{chol}(P)x) \leq 1$
Useful Resources

- http://cvxr.com/cvx/