Outline

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- Definition
- Properties
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- Dual CP
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SOCP Duality

SDP Duality
What’s the dual to a conic program?

Recall the LP duality:

\[
\begin{align*}
\min \quad & c^T x \\
\text{(LP)} \quad \text{s.t.} \quad & Ax \geq b \\
\max \quad & b^T y \\
\text{(LD)} \quad \text{s.t.} \quad & A^T y = c \\
\end{align*}
\]

\[y \geq 0\]

Now consider the conic program

\[
\begin{align*}
\min \quad & c^T x \\
\text{(CP)} \quad \text{s.t.} \quad & Ax \geq_K b \\
\text{(CD)} \quad ?
\end{align*}
\]

Q. Now that \(Ax \geq_K b \Rightarrow y^T(Ax) \geq y^T b\) for which \(y\)?
Definition. The dual cone of a nonempty cone $\mathcal{K}$ is

$$\mathcal{K}_* = \left\{ y : y^T x \geq 0, \forall x \in \mathcal{K} \right\}$$

Remark. Dual cone is always a closed cone.
Properties of Dual Cone

**Proposition.** Let $\mathcal{K}$ be a closed cone and $\mathcal{K}^*$ be its dual.

(a) $(\mathcal{K}^*)^* = \mathcal{K}$

(b) $\mathcal{K}$ is pointed iff $\mathcal{K}^*$ has non-empty interior

(c) $\mathcal{K}$ is a regular cone iff $\mathcal{K}^*$ is a regular cone

*Proof:* Self-exercise.
Self-dual Cone

**Definition.** If $\mathcal{K} = \mathcal{K}^*$, we call it a self-dual cone.

**Remark.** Nonnegative orthant, second order cone, and positive semidefinite cone are all self-dual:

- $(\mathbb{R}_+^m)^* = \mathbb{R}_+^m$
- $(L^n)^* = L^n$
- $(S_+^n)^* = S_+^n$
Self-dual Cone

**Proposition.** $L^n$ is self-dual, i.e. $(L^n)^* = L^n$.

**Proof**

(i) $L^n \subseteq (L^n)^*$: Suppose $y \in L^n$, we show that $\forall x \in L^n$,

$$y^T x = y_1x_1 + \ldots + y_nx_n \geq -\sqrt{\sum_{i=1}^{n-1} y_i^2} \sqrt{\sum_{i=1}^{n-1} x_i^2} + y_nx_n \geq 0$$

due to Cauchy-Schwarz inequality.

(ii) $(L^n)^* \subseteq L^n$: Suppose $y \in (L^n)^*$, we have $y^T x \geq 0$, $\forall x \in L^n$

If $(y_1, \ldots, y_{n-1}) = 0$, let $x = [0, \ldots, 0, 1] \in L^n$, we get

$$y^T x = y_n \geq 0, \Rightarrow y \in L^n.$$  

Otherwise, let $x = [-y_1, \ldots, -y_{n-1}, \sqrt{\sum_{i=1}^{n-1} y_i^2}] \in L^n$,

$$y^T x = -\sum_{i=1}^{n-1} y_i^2 + y_n\sqrt{\sum_{i=1}^{n-1} y_i^2} \geq 0 \Rightarrow y \in L^n.$$
Dual of Conic Program

Primal Conic Program:

\[
\begin{align*}
\text{min} & \quad c^T x \\
\text{s.t.} & \quad Ax \succeq_K b
\end{align*}
\]

(DP)

Dual Conic Program:

\[
\begin{align*}
\text{max} & \quad b^T y \\
\text{s.t.} & \quad A^T y = c \\
& \quad y \succeq_K 0
\end{align*}
\]

(CD)
Conic Duality

Theorem.

- **(Weak Conic Duality):** \( \text{Opt}(CD) \leq \text{Opt}(CP) \)
- **(Strong Conic Duality):** If \((CP)\) is bounded below and strictly feasible, i.e.,

\[
\exists x_0, \text{ s.t. } Ax_0 \succ K b,
\]

then \((CD)\) is solvable and \(\text{Opt}(CD) = \text{Opt}(CP)\).

Corollary. If \((CD)\) is bounded above and strictly feasible,

i.e. \(\exists y \succ K^* 0, \text{ s.t. } A^T y = c\)

then \((CP)\) is solvable and \(\text{Opt}(CD) = \text{Opt}(CP)\).
Proof of Conic Duality

Denote $p^* = \text{Opt}(CP)$.
Sufficient to show that $\exists y^*$ feasible to (CD), s.t., $b^T y^* \geq p^*$.
When $c = 0$, simply set $y^* = 0$. Now consider $c \neq 0$. Define

$$M = \left\{ Ax - b : c^T x \leq p^* \right\}.$$ 

- $M \cap \text{int}(K) = \emptyset$ (why?)
- By separation theorem, $\exists y \neq 0$, s.t.
  $$\sup_{z \in M} y^T z \leq \inf_{z \in \text{int}(K)} y^T z$$
- It must hold that $y \in K_*$ and $\sup_{x : c^T x \leq p^*} y^T (Ax - b) \leq 0$.
- Hence, $\lambda c = A^T y$ for some $\lambda \geq 0$.
- By strictly feasibility of (CP), we further have $\lambda > 0$ (why?).
- Setting $y^* = \frac{\lambda}{\lambda}$, we have $y^* \in K_*$, $A^T y^* = c$ and $p^* \leq b^T y^*$. 
Optimality Conditions

**Theorem.** Suppose at least one of \((CP)\) and \((CD)\) is bounded and strictly feasible, then the feasible primal-dual pair \((x^*, y^*)\) is a pair of optimal primal-dual solutions iff

- (Zero duality gap) \(c^T x^* - b^T y^* = 0\)
- (Complementary slackness) \((Ax^* - b)^T y^* = 0\)

Observe that

\[
c^T x^* - b^T y^* = c^T x^* - \text{Opt}(CP) \geq 0
\]

\[
+ \text{Opt}(CD) - b^T y^* \geq 0
\]

\[
+ \text{Opt}(CP) - \text{Opt}(CD) \geq 0
\]
Discussions

- In the case of \( LP \), strict feasibility is not required for strong duality nor solvability of the program.

- In general case of \( CP \), strict feasibility is required.
A conic problem can be strictly feasible and bounded, but NOT solvable.

\[
\begin{align*}
\min_{x_1, x_2} & \quad x_1 \\
\text{s.t.} & \quad \begin{bmatrix} x_1 - x_2 \\ 1 \\ x_1 + x_2 \end{bmatrix} \succeq_{L^3} 0 \\
\end{align*}
\]

\[
\begin{align*}
\min_{x_1, x_2} & \quad x_1 \\
\text{s.t.} & \quad 4x_1 x_2 \geq 1 \\
& \quad x_1 + x_2 > 0
\end{align*}
\]
A conic problem can be solvable yet not strictly feasible, and the dual is infeasible.

\[
\begin{align*}
\min_{x_1, x_2} & \quad x_2 \\
\text{s.t.} & \quad \begin{bmatrix} x_1 \\ x_2 \\ x_1 \end{bmatrix} \succeq_{L^3} 0 \\
& \quad \iff \\
& \quad \max_{\lambda} \quad 0 \\
& \quad \text{s.t.} \quad \begin{bmatrix} \lambda_1 + \lambda_3 \\ \lambda_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
& \quad \lambda \succeq_{L^3} 0
\end{align*}
\]
SOCP Duality

Primal SOCP:

\[
\begin{align*}
\min_x & \quad c^T x \\
\text{s.t.} & \quad \|A_i x - b_i\|_2 \leq d_i^T x - e_i, \quad i = 1, \ldots, m \quad (\text{SOCP–P})
\end{align*}
\]

Dual SOCP:

\[
\begin{align*}
\max_{\lambda \in \mathbb{R}^m} & \quad \sum_{i=1}^m b_i^T u_i + e^T \lambda \\
\text{s.t.} & \quad \sum_{i=1}^m (A_i^T u_i + d_i \lambda_i) = c \quad (\text{SOCP–D}) \\
& \quad \|u_i\|_2 \leq \lambda_i, \quad i = 1, \ldots, m
\end{align*}
\]
SDP Duality

Primal SDP:

\[
\begin{align*}
\min & \quad c^T x \\
\text{s.t.} & \quad A x - B = \sum_{i=1}^{n} x_i A_i - B \succeq 0 \quad (\text{SDP–P})
\end{align*}
\]

Dual SDP:

\[
\begin{align*}
\max_Y & \quad \text{tr}(B Y) \\
\text{s.t.} & \quad \text{tr}(A_i Y) = c_i \quad i = 1, \ldots, n \quad (\text{SDP–D})
\end{align*}
\]

\[
Y \succeq 0
\]
SDP Optimality Conditions

\[
\begin{align*}
\min & \quad c^T x \\
\text{s.t.} & \quad \sum_{i=1}^{n} x_i A_i - B \succeq 0
\end{align*}
\]

\[
\begin{align*}
\max & \quad \text{tr}(BY) \\
\text{s.t.} & \quad \text{tr}(A_i Y) = c_i, \ i = 1, \ldots, n
\end{align*}
\]

Remark. \((x^*, Y^*)\) is optimal primal-dual pair iff

1. \(\sum_{i=1}^{n} x_i^* A_i \succeq B\) (primal feasibility)
2. \(Y^* \succeq 0, \text{tr}(A_i Y^*) = c_i, \ i = 1, \ldots, m\) (dual feasibility)
3. \(Y^*(\sum_{i=1}^{n} x_i^* A_i - B) = 0\) (complementary slackness)
Application of SDP Duality

**Example.** Use SDP duality to show that for any $B \in S^n_+$:

$$\lambda_{\text{max}}(B) = \max_{x \in \mathbb{R}^n} \left\{ x^T B x : \|x\|_2 = 1 \right\}$$

$$\begin{align*}
\max_x & \quad \text{tr}(Bxx^T) \\ 
\text{s.t.} & \quad \text{tr}(xx^T) = 1 \quad (P)
\end{align*}$$

$$\begin{align*}
\max_X & \quad \text{tr}(BX) \\ 
\text{s.t.} & \quad \text{tr}(X) = 1 \quad (P') \\
& \quad X \succeq 0
\end{align*}$$

$$\Leftrightarrow$$

$$(P) \equiv (P')$$, why?

$$\begin{align*}
\min_x & \quad \lambda \\ 
\text{s.t.} & \quad \lambda I - B \succeq 0 \quad (D)
\end{align*}$$
SDP Relaxation of Nonconvex QCQP

Quadratic constrained quadratic programming:

\[
\begin{align*}
\min & \quad x^T Q_0 x + 2q_0^T x + c_0 \\
\text{s.t.} & \quad x_i^T Q_i x_i + 2q_i^T x + c_i \leq 0, \quad 1 \leq i \leq m
\end{align*}
\]

(QCQP)

Rank-1 reformulation:

\[
\begin{align*}
\min_{x,X} & \quad \text{tr}(A_0 X) \\
\text{s.t.} & \quad \text{tr}(A_i X) \leq 0, \quad 1 \leq i \leq m
\end{align*}
\]

(QCQP')

\[
X = \begin{bmatrix}
xx^T & x \\
x^T & 1
\end{bmatrix}
\]

Here \( A_i = \begin{bmatrix} Q_i & q_i \\ q_i^T & c_i \end{bmatrix}, \quad i = 0, 1, \ldots, m \)
SDP Relaxation of Nonconvex QCQP

SDP relaxation:

$$\begin{align*}
\min_X & \quad \text{tr}(A_0X) \\
n & \quad \text{tr}(A_iX) \leq 0, \ 1 \leq i \leq m \quad \text{(SDP-r)} \\
X & \succeq 0 \\
X_{n+1,n+1} & = 1
\end{align*}$$

Dual of SDP relaxation:

$$\begin{align*}
\max_{\lambda \geq 0, t} & \quad t \\
\text{s.t.} & \quad A_0 + \sum \lambda_i A_i - \begin{bmatrix} 0 & 0 \\ 0 & t \end{bmatrix} \succeq 0 \quad \text{(SDP-d)}
\end{align*}$$

Remark. Opt(SDP-d) \leq Opt(SDP-r) \leq Opt(QCQP)
References

- Ben-Tal & Nemirovski (2013), Chapters 1 - 3