Outline

Warm-up

Topology Review

Convex Sets
  Definitions
  Convex Hull
  Examples
  Calculus of Convexity
  Topological Properties
  Representation Theorem
Which set is different from others?

Figure: Four sets
Which set is different from others?

Figure: Four sets
Interior, Closure, Boundary

**Definition.** Let \( X \) be a nonempty set in \( \mathbb{R}^n \).

- A point \( x_0 \) is called an *interior point* if \( \exists r > 0 \), such that \( B(x_0, r) := \{ x : \|x - x_0\|_2 \leq r \} \subseteq X \).
- A point \( x_0 \) is called a *limit point* if \( \exists \{ x_n \} \subseteq X \), such that \( x_n \rightarrow x_0 \) as \( n \rightarrow \infty \).

**Definition.**

- Interior: \( \text{int}(X) = \) the set of all interior point of \( X \).
- Closure: \( \text{cl}(X) = \) the set of all limit points of \( X \).
- Boundary: \( \partial(X) = \text{cl}(X)/\text{int}(X) \)

Q. Let \( X = \) irrationals on \([0, 1]\). What are \( \text{int}(X) \) and \( \text{cl}(X) \)?
Open and Closed Sets

Definition.
- $X$ is **closed** if $\text{cl}(X) = X$;
- $X$ is **open** if $\text{int}(X) = X$.

Fact.
- $\text{int}(X) \subseteq X \subseteq \text{cl}(X)$;
- $X$ is closed iff $X^c = \mathbb{R}^n / X$ is open;
- $\bigcap_{\alpha \in \mathcal{A}} X_\alpha$ is closed if $X_\alpha$ is closed for all $\alpha \in \mathcal{A}$.
- $\bigcup_{i=1}^n X_i$ is closed if $X_i$ is closed for $i = 1, \ldots, n$.

Q. If $X_1, X_2$ are closed, is $X_1 + X_2$ closed?
Convex Set

Definition

- A set \( X \subseteq \mathbb{R}^n \) is convex if
  \[
  x, y \in X, \lambda \in [0, 1] \Rightarrow \lambda x + (1 - \lambda)y \in X.
  \]

- In another word, the line segment \([x, y]\) that connects any two elements \(x, y\) lies entirely in the set.

\(\text{Figure: Examples of convex and non-convex sets}\)
Definition. Given any elements $x_1, \ldots, x_k$, the combination $\lambda_1 x_1 + \ldots + \lambda_k x_k$ is called

- **Convex**: if $\lambda_i \geq 0$, $i = 1, \ldots, k$ and $\lambda_1 + \ldots + \lambda_k = 1$;
- **Conic**: if $\lambda_i \geq 0$, $i = 1, \ldots, k$;
- **Affine**: if $\lambda_1 + \ldots + \lambda_k = 1$;
- **Linear**: if $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, k$. 
Convex Sets, Cones, Affine and Linear Subspaces

Definition.

- A set is **convex** if all convex combinations of its elements are in the set;
- A set is a **convex cone** if all conic combinations of its elements are in the set;
- A set is a **affine subspace** if all affine combinations of its elements are in the set;
- A set is a **linear subspace** if all linear combinations of its elements are in the set.

Clearly, a linear subspace is always a convex cone; a convex cone is always a convex set.

**Note:** Cones vs. Convex cones.
Definiton. Given any set $X$, we define

- **Convex hull** of $X$:
  \[ \text{Conv}(X) = \text{set of all convex combinations of points in } X. \]

- **Conic hull** of $X$:
  \[ \text{Cone}(X) = \text{set of all conic combinations of points in } X. \]

- **Affine hull** of $X$:
  \[ \text{Aff}(X) = \text{set of all affine combinations of points in } X. \]

Figure: Examples of convex hulls
Properties of Convex Sets

Proposition.

1. A convex hull is always convex.
2. If \( X \) is convex, then \( \text{Conv}(X) = X \).
3. For any set \( X \), \( \text{Conv}(X) \) is the smallest convex set that contains \( X \).
Examples of Convex Sets

Example 1. Simple sets:

- **Hyperplane**: \( \{ x \in \mathbb{R}^n : a^T x = b \} \)
- **Halfspace**: \( \{ x \in \mathbb{R}^n : a^T x \leq b \} \)
- **Affine space**: \( \{ x \in \mathbb{R}^n : Ax = b \} \)
- **Polyhedron**: \( \{ x \in \mathbb{R}^n : Ax \leq b \} \)
- **Simplex**: \( \{ x \in \mathbb{R}^n : x \geq 0, \sum_{i=1}^{n} x_i = 1 \} \).

Example 2. Euclidean balls:

\[ \{ x \in \mathbb{R}^n : \| x - a \|_2 \leq r \} \]

Example 3. Ellipsoid:

\[ \{ x \in \mathbb{R}^n : (x - a)^T Q(x - a) \leq r^2 \} \]

where \( Q \succ 0 \) and is symmetric.
Examples of Convex Cones

Example 1. Positive Orthant:
\[ \{ x \in \mathbb{R}^n : x \geq 0 \} \]

Example 2. Norm cones:
\[ \{(x, t) \in \mathbb{R}^{n+1} : \| x \|_2 \leq t \} \]

Example 3. Positive semidefinite matrices:
\[ \mathbb{S}_+^n := \{ X \in \mathbb{S}^n : X \succeq 0 \} \]
Operations that Preserves Convexity

**Intersection**
- If $X_\alpha, \alpha \in \mathcal{A}$ are convex sets, then so is
  $$\bigcap_{\alpha \in \mathcal{A}} X_\alpha.$$ 

**Cartesian product:**
- If $X_i \subseteq \mathbb{R}^{n_i}, i = 1, \ldots, k$ are convex, then so is
  $$X_1 \times \cdots \times X_k.$$ 

**Weighted summation:**
- If $X_i \subseteq \mathbb{R}^n, i = 1, \ldots, k$ convex, then so is
  $$\alpha_1 X_1 + \cdots + \alpha_k X_k.$$
Operations that Preserves Convexity

Affine image:

- If \( X \subseteq \mathbb{R}^n \) is a convex set and \( \mathcal{A}(x) : x \mapsto Ax + b \) is an affine mapping from \( \mathbb{R}^n \) to \( \mathbb{R}^k \), then so is

\[
\mathcal{A}(X) := \{Ax + b : x \in X\}.
\]

**Proof:**

Let \( y_1, y_2 \in \mathcal{A}(X) \Rightarrow \exists x_1, x_2 \in X \) such that 
\( y_1 = Ax_1 + b \) and \( y_2 = Ax_2 + b \). For \( \lambda \in [0, 1] \),

\[
\lambda y_1 + (1 - \lambda)y_2 = A(\lambda x_1 + (1 - \lambda)x_2) + b \in \mathcal{A}(X)
\]

because \( \lambda x_1 + (1 - \lambda)x_2 \in X \).
Operations that Preserves Convexity

Inverse affine image:

- If $X \subseteq \mathbb{R}^n$ is a convex set and $\mathcal{A}(y) : y \mapsto Ay + b$ is an affine mapping from $\mathbb{R}^k$ to $\mathbb{R}^n$, then so is

  $$\mathcal{A}^{-1}(X) := \{y : Ay + b \in X\}.$$

- Proof: self-exercise.

Example. The solution set of linear matrix inequality:

$$\{x | x_1 A_1 + \cdots + x_k A_k \preceq B\}$$

where $A_i, B$ are positive semidefinite matrices.
Nice Properties of Convex Sets

**Proposition.** Let $X$ be convex with nonempty interior. Then

- If $x_0 \in \text{int}(X)$ and $x \in \text{cl}(X)$, then $[x_0, x) \in \text{int}(X)$.
- Moreover, $\text{int}(X)$ is dense in $\text{cl}(X)$.

**Remark.** In general, $\text{int}(X)$ and $\text{cl}(X)$ can differ dramatically.

- If $X = \text{irrationals}$ on $[0, 1]$, $\text{int}(X) = \emptyset$, $\text{cl}(X) = [0, 1]$.

**Q.** What happens if $X$ is convex but $\text{int}(X) = \emptyset$?
Nice Properties of Convex Sets

**Definition.** (Relative Interior and Dimension)

- \( \text{rint}(X) = \{ x : \exists r > 0, \text{ s.t. } B(x, r) \cap \text{Aff}(X) \subseteq X \} \)
- \( \text{dim}(X) = \text{dim}(\text{Aff}(X)) \)

**Fact.** For a convex and nonempty set, \( \text{rint}(X) \neq \emptyset \).

**Proposition.** Let \( X \) be a nonempty convex set. Then

a) \( \text{int}(X), \text{cl}(X), \text{rint}(X) \) are convex

b) \( x_0 \in \text{rint}(X), x \in \text{cl}(X) \Rightarrow [x_0, x] \in \text{rint}(X), \forall \lambda \in (0, 1] \)

c) \( \text{cl}(\text{rint}(X)) = \text{cl}(X) \)

d) \( \text{rint}(\text{cl}(X)) = \text{rint}(X) \)
Question

Suppose there are 100 different kinds of herbal tea, everyone of them is a blend of 25 herbs. Donald wants a particular mixture of all herbal teas with equal proportions. What’s the least number of teas he should buy?
Carathéodory Representation Theorem

**Theorem.** Let $X \subseteq \mathbb{R}^n$ be non empty and $\dim(X) = d \leq n$. Every point $x \in \text{Conv}(X)$ is a convex combination of at most $(d + 1)$ points, i.e.

$$\text{Conv}(X) = \left\{ \sum_{i=1}^{d+1} \lambda_i x_i : x_i \in X, \lambda_i \geq 0, \sum_{i=1}^{d+1} \lambda_i = 1 \right\}.$$

**Proof:** Suppose the minimal representation of $x \in \text{Conv}(X)$ has $m \geq d + 1$ terms, $x = \sum_{i=1}^{m} \alpha_i x_i$, where $\alpha_i \geq 0, \sum_{i=1}^{m} \alpha_i = 1$. The system of linear equations

$$\begin{cases}
\sum_{i=1}^{m} \delta_i x_i = 0 \\
\sum_{i=1}^{m} \delta_i = 0
\end{cases}$$

has non trivial solution.

Rewrite $x = \sum_{i=1}^{m} (\alpha_i - t\delta_i) x_i$. Let $\lambda_i(t) = (\alpha_i - t\delta_i), \forall i$, we have $\sum \lambda_i(t) = 1$. Let $t_* = \min \left\{ \frac{\alpha_i}{\delta_i}, \delta_i > 0 \right\} := \frac{\alpha_j}{\delta_j}$, then $\lambda_i(t_*) > 0, \forall i \neq j$ and $\lambda_j(t_*) = 0$. This leads to a smaller representation of $x$. 


References

- Boyd & Vandenberghe, Chapter 2.1-2.3
- Ben-Tal & Nemirovski, Chapter 1.1