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In this lecture, we cover the following topics

- Topology review
- Convex sets (convex/conic/affine hulls)
- Examples of convex sets
- Calculus of convex sets
- Some nice topological properties of convex sets.

### 1.1 Topology Review

Let $X$ be a nonempty set in $\mathbb{R}^n$. A point $x_0$ is called an **interior point** if $X$ contains a small ball around $x_0$, i.e. $\exists r > 0$, such that $B(x_0, r) := \{ x : \|x - x_0\|_2 \leq r \} \subseteq X$. A point $x_0$ is called a **limit point** if there exists a convergent sequence in $X$ that converges to $x_0$, i.e. $\exists \{x_n\} \subseteq X$, such that $x_n \rightarrow x_0$ as $n \rightarrow \infty$.

- The interior of $X$, denoted as $\text{int}(X)$, is the set of all interior point of $X$.
- The closure of $X$, denoted as $\text{cl}(X)$, is the set of all limit points of $X$.
- The boundary of $X$, denoted as $\partial(X) = \text{cl}(X)/\text{int}(X)$, is the set of points that belongs to the closure but not in the interior.

$X$ is **closed** if $\text{cl}(X) = X$; $X$ is **open** if $\text{int}(X) = X$. Here are some basic facts:

- $\text{int}(X) \subseteq X \subseteq \text{cl}(X)$;
- A set $X$ is closed if and only if its complement $X^c = \mathbb{R}^n/X$ is open;
- The intersection of arbitrary number of closed sets is closed, i.e., $\cap_{\alpha \in \mathcal{A}}X_\alpha$ is closed if $X_\alpha$ is closed for all $\alpha \in \mathcal{A}$.
- The union of finite number of closed sets is closed, i.e., $\cup_{i=1}^nX_i$ is closed if $X_i$ is closed for $i = 1, \ldots, n$. 


1.2 Convex Sets

Definition 1.1 (Convex set) A set $X \subseteq \mathbb{R}^n$ is convex if $\forall x, y \in X, \lambda x + (1 - \lambda)y \in X$ for any $\lambda \in [0, 1]$.

In another word, the line segment that connects any two elements lies entirely in the set.

![Diagram of convex and non-convex sets](image)

(a) convex set  
(b) non-convex set

Figure 1.1: Examples of convex and non-convex sets

Given any elements $x_1, \ldots, x_k$, the combination $\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_k x_k$ is called

- **Convex**: if $\lambda_i \geq 0, i = 1, \ldots, k$ and $\lambda_1 + \lambda_2 + \ldots + \lambda_k = 1$;
- **Conic**: if $\lambda_i \geq 0, i = 1, \ldots, k$;
- **Affine**: if $\lambda_1 + \lambda_2 + \ldots + \lambda_k = 1$;
- **Linear**: if $\lambda_i \in \mathbb{R}, i = 1, \ldots, k$.

Consequently, we have

- A set is convex if all convex combinations of its elements are in the set;
- A set is a convex cone if all conic combinations of its elements are in the set;
- A set is an affine subspace if all affine combinations of its elements are in the set;
- A set is a linear subspace if all linear combinations of its elements are in the set.

Clearly, a linear subspace is always a convex cone; a convex cone is always a convex set.
**Definition 1.2 (Convex hull)** A convex hull of a set \( X \subseteq \mathbb{R}^n \) is the set of all convex combination of its elements, denoted as

\[
\text{Conv}(X) = \left\{ \sum_{i=1}^{k} \lambda_i x_i : k \in \mathbb{N}, \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1, x_i \in X, \forall i = 1, \ldots, k \right\}.
\]

Similarly, one can define the conic hull and affine hull of a set.

\[
\text{Cone}(X) = \left\{ \sum_{i=1}^{k} \lambda_i x_i : k \in \mathbb{N}, x_i \in X, \lambda_i \geq 0, \forall i = 1, \ldots, k \right\}.
\]

\[
\text{Aff}(X) = \left\{ \sum_{i=1}^{k} \lambda_i x_i : k \in \mathbb{N}, x_i \in X, \sum_{i=1}^{k} \lambda_i = 1, \forall i = 1, \ldots, k \right\}.
\]

**Proposition 1.3** We have the following

1. A convex hull is always convex.
2. If \( X \) is convex, then \( \text{conv}(X) = X \).
3. For any set \( X \), \( \text{conv}(X) \) is the smallest convex set that contains \( X \).

**Proof:**

1. By definition, for any \( x, y \in \text{Conv}(X) \), we can write \( x = \sum_i \lambda_i x_i \) and \( y = \sum_i \mu_i x_i \) where \( \lambda_i, \mu_i \geq 0 \) and \( \sum_i \lambda_i = \sum_i \mu_i = 1 \). Hence, for any \( \alpha \in [0, 1] \), we have

\[
\alpha x + (1 - \alpha) y = \alpha \sum_i \lambda_i x_i + (1 - \alpha) \sum_i \mu_i x_i = \sum_i \xi_i x_i
\]

where \( \xi_i = \alpha \lambda_i + (1 - \alpha) \mu_i, \forall i \). Note that \( \xi_i \geq 0 \) and \( \sum_i \xi_i = \alpha \sum_i \lambda_i + (1 - \alpha) \sum_i \mu_i = 1 \). Therefore, \( \alpha x + (1 - \alpha) y \in \text{Conv}(X) \). Hence, \( \text{Conv}(X) \) is convex.
2. First of all, based on definition of convex hull, it is straightforward to see that $X \subseteq \text{Conv}(X)$. Next, we show that $\text{Conv}(X) \subseteq X$ by induction on $k$. The baseline is when $k = 1$, which is trivial. Now assuming that any convex combination with $k$ entries is in $X$, we want to show that any convex combination of $k + 1$ entries is still in $X$. Consider the convex combination below given by $\lambda_1, \ldots, \lambda_{k+1}$ with $\lambda_i \geq 0, i = 1, \ldots, k + 1$ and $\sum_{i=1}^{k+1} \lambda_i = 1$.

$$\lambda_1 x_1 + \ldots + \lambda_{k+1} x_{k+1} = (1 - \lambda_{k+1}) \left( \frac{\lambda_1}{1 - \lambda_{k+1}} x_1 + \ldots + \frac{\lambda_k}{1 - \lambda_{k+1}} x_k \right) + \lambda_{k+1} x_{k+1}$$

Based on induction, we can see that $z \in X$ since $z$ is a convex combination of $k$ entries in $X$. By convexity of $X$, we further have $\lambda_1 x_1 + \ldots + \lambda_{k+1} x_{k+1} \in X$.

3. Suppose $Y$ is convex and $Y \supseteq X$, we want to show that $Y \supseteq \text{Conv}(X)$. From previous argument, if $Y$ contains $X$, then $Y$ should contain all convex combinations of $X$, i.e. $Y \supseteq \text{Conv}(X)$.

Examples of Convex Sets

Example 1. Some simple convex sets:

- **Hyperplane:** $\{ x \in \mathbb{R}^n : a^T x = b \}$
- **Halfspace:** $\{ x \in \mathbb{R}^n : a^T x \leq b \}$
- **Affine space:** $\{ x \in \mathbb{R}^n : Ax = b \}$
- **Polyhedron:** $\{ x \in \mathbb{R}^n : Ax \leq b \}$
- **Simplex:** $\{ x \in \mathbb{R}^n : x \geq 0, \sum_{i=1}^{n} x_i = 1 \} = \text{conv}(e_1, \ldots, e_n)$.

Example 2. **Euclidean balls:**

$$\{ x \in \mathbb{R}^n : \| x \|_2 \leq r \}$$

where $\| \cdot \|_2$ is the Euclidean norm defined on $\mathbb{R}^n$.

Example 3. **Ellipsoid:**

$$\{ x \in \mathbb{R}^n : (x - a)^T Q (x - a) \leq r^2 \}$$

where $Q \succ 0$ and is symmetric.
1.3 Calculus of Convex Sets

The following operators preserve the convexity of sets, which can be easily verified based on the definition.

1. **Intersection**: If $X_\alpha, \alpha \in \mathcal{A}$ are convex sets, then

$$\bigcap_{\alpha \in \mathcal{A}} X_\alpha$$

is also a convex set.

2. **Direct product**: If $X_i \subseteq \mathbb{R}^n, i = 1, \ldots, k$ are convex sets, then

$$X_1 \times \cdots \times X_k := \{(x_1, \ldots, x_k) : x_i \in X_i, i = 1, \ldots, k\}$$

is also a convex set.

3. **Weighted summation**: If $X_i \subseteq \mathbb{R}^n, i = 1, \ldots, k$ are convex sets, then

$$\alpha_1 X_1 + \cdots + \alpha_k X_k := \{\alpha_1 x_1 + \cdots + \alpha_k x_k : x_i \in X_i, i = 1, \ldots, k\}$$

is also a convex set.

4. **Affine image**: If $X \subseteq \mathbb{R}^n$ is a convex set and $A(x) : x \mapsto Ax + b$ is an affine mapping from $\mathbb{R}^n$ to $\mathbb{R}^k$, then

$$A(X) := \{Ax + b : x \in X\}$$

is also a convex set.

5. **Inverse affine image**: If $X \subseteq \mathbb{R}^n$ is a convex set and $A(y) : y \mapsto Ay + b$ is an affine mapping from $\mathbb{R}^k$ to $\mathbb{R}^n$, then

$$A^{-1}(X) := \{y : Ay + b \in X\}$$

is also a convex set.

**Proof:**

1. Let $x, y \in \bigcap_{\alpha \in \mathcal{A}} X_\alpha$, then $x, y \in X_\alpha, \forall \alpha \in \mathcal{A}$. Since $X_\alpha$ is convex, for any $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda)y \in X_\alpha, \forall \alpha \in \mathcal{A}$. Hence, $\lambda x + (1 - \lambda)y \in \bigcap_{\alpha \in \mathcal{A}} X_\alpha$.

2. Let $x = (x_1, \ldots, x_k) \in X_1 \times \cdots \times X_k, y = (y_1, \ldots, y_k) \in X_1 \times \cdots \times X_k$. Since $X_i$ is convex, for $\lambda \in [0, 1], \lambda x_i + (1 - \lambda)y_i \in X_i, \forall i = 1, \ldots, k$. Hence

$$\lambda x + (1 - \lambda)y = (\lambda x_1 + (1 - \lambda)y_1, \ldots, \lambda x_k + (1 - \lambda)y_k) \in X_1 \times \cdots \times X_k.$$ 

3. Let $x, y \in \alpha_1 X_1 + \cdots + \alpha_k X_k$, by definition, there exists $x_i, y_i \in X_i, i = 1, \ldots, k$, such that

$$x = \alpha_1 x_1 + \cdots + \alpha_k x_k, y = \alpha_1 y_1 + \cdots + \alpha_k y_k.$$ 

Hence, for all $\lambda \in [0, 1]$

$$\lambda x + (1 - \lambda)y = \lambda x_1 + \cdots + \alpha_k z_k \in \alpha_1 X_1 + \cdots + \alpha_k X_k$$

because $z_i = \lambda x_i + (1 - \lambda)y_i \in X_i, \forall i = 1, \ldots, k.$
4. Let \( y_1, y_2 \in \mathcal{A}(X) \), then there exists \( x_1, x_2 \in X \) such that \( y_1 = Ax_1 + b \) and \( y_2 = Ax_2 + b \). Therefore, for any \( \lambda \in [0, 1] \), we have \( \lambda y_1 + (1 - \lambda)y_2 = A(\lambda x_1 + (1 - \lambda)x_2) + b \in \mathcal{A}(X) \) because \( \lambda x_1 + (1 - \lambda)x_2 \in X \).

5. Let \( y_1, y_2 \in \mathcal{A}^{-1}(X) \), then there exits \( x_1, x_2 \in X \) such that \( x_1 = Ay_1 + b \) and \( x_2 = Ay_2 + b \). Therefore, for any \( \lambda \in [0, 1] \), we have \( A(\lambda y_1 + (1 - \lambda)y_2) + b = \lambda x_1 + (1 - \lambda)x_2 \in X \), this implies that \( \lambda y_1 + (1 - \lambda)y_2 \in \mathcal{A}^{-1}(X) \).

\[ \blacksquare \]

1.4 Nice Topological Properties of Convex Sets

Convex sets are special because of their nice geometric properties.

**Proposition 1.4** If \( X \) be a convex set with nonempty interior, then \( \text{int}(X) \) is dense in \( \text{cl}(X) \).

**Proof:** Let \( x_0 \in \text{int}(X) \) and \( x \in \text{cl}(X) \). We can construct a convergence sequence \( y_n = \frac{1}{n}x_0 + (1 - \frac{1}{n})x \) such that \( y_n \to x \). We only need to show that \( y_n \in \text{int}(X) \). Therefore, it suffices to prove the following claim:

**Claim 1.5** If \( x_0 \in \text{int}(X) \) and \( x \in \text{cl}(X) \), then \( [x_0, x] \in \text{int}(X) \), namely, for any \( \alpha \in [0, 1) \), the point \( z := \alpha x_0 + (1 - \alpha) x \in \text{int}(X) \).

This can be proved as follows. Since \( x_0 \in \text{int}(X) \), there exits \( r > 0 \) such that \( B(x_0, r) \subseteq X \). Since \( x \in \text{cl}(X) \), there exits a sequence \( \{x_n\} \subseteq X \) such that \( x_n \to x \). Let \( z_n = \alpha x_0 + (1 - \alpha)x_n \), then \( z_n \to z \). When \( n \) is large enough, \( \|z_n - z\|_2 \leq \frac{\alpha r}{2} \). Since \( B(x_0, r) \subseteq X \) and \( x_n \in X \), then \( B(z_n, \alpha r) = \alpha B(x_0, r) + (1 - \alpha)x_n \subseteq X \). Hence, \( B(z, \frac{\alpha r}{2}) \subseteq B(z_n, \alpha r) \subseteq X \). This is because for any \( z' \in B(z, \frac{\alpha r}{2}) \), \( \|z' - z\| \leq \frac{\alpha r}{2} \),

\[
\|z' - z\|_2 \leq \|z' - z\| + \|z_n - z\|_2 \leq \frac{\alpha r}{2} + \frac{\alpha r}{2} = \alpha r.
\]

\[ \blacksquare \]

**Remark.** Note that in general, for any set \( X \), \( \text{int}(X) \subseteq X \subseteq \text{cl}(X) \), but \( \text{int}(X) \) and \( \text{cl}(X) \) can differ dramatically. For instance, let \( X \) be the set of all irrational numbers in \((0, 1)\), then \( \text{int}(X) = \emptyset \), \( \text{cl}(X) = [0, 1] \). The proposition implies that a convex set is perfectly well characterized by its closure or interior if nonempty.
In this lecture, we cover the following topics

- Nice topological properties (cont’d)
- Representation Theorem (Caratheodory)
- Radon, Helley Theorems

2.1 Recall

- A set $X$ is convex if $\forall x, y \in X, \lambda x + (1 - \lambda)y \in X$ for any $\lambda \in [0, 1]$.
- Convex Hull:
  \[
  \text{Conv}(X) = \left\{ \sum_{i=1}^{k} \lambda_i x_i : k \in \mathbb{N}, \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1, x_i \in X, \forall i = 1, \ldots, k \right\}.
  \]
  Note that the convex hull of $X$ is the smallest convex set that contains $X$.
- Affine Hull:
  \[
  \text{aff}(X) = \left\{ \sum_{i=1}^{k} \lambda_i x_i : k \in \mathbb{N}, x_i \in X, \sum_{i=1}^{k} \lambda_i = 1, \forall i = 1, \ldots, k \right\}.
  \]
  Note that the affine hull of $X$ is the smallest affine subspace that contains $X$. An affine space $M$ is a shifted linear space, i.e. $M = \{a\} + L$, e.g. $\{x : Ax = b\} = x_0 + \{x : Ax = 0\}$ where $x_0$ is such that $Ax_0 = b$. The dimension of $X$: $\dim(X) = \dim(\text{aff}(X))$
- If $X \in \mathbb{R}^n$ is convex with non-empty interior, then
  \[
  \forall x_0 \in \text{int}(X), x \in \text{cl}(X) \Rightarrow \lambda x_0 + (1 - \lambda)x \in \text{int}(X).
  \]
  Moreover, the interior of $X$ if nonempty, is dense in $\text{cl}(X)$, i.e., $\text{cl}(\text{int}X) = \text{cl}(X)$

Questions: What if $\text{int}(X) = \emptyset$? For example, $X = \text{Conv}(e_1, e_2, e_3) = \{(x_1, x_2, x_3) : x_1, x_2, x_3 \geq 0, x_1 + x_2 + x_3 = 1\}$, $\text{int}(X) = \emptyset$, $\text{aff}(X) = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 1\}$, $\dim(X) = 2$. 

2-1
Definition 2.1 (Relative Interior) \( \text{rint}(X) = \{ x : \exists r > 0, \text{s.t.} B(x, r) \cap \text{Aff}(X) \subseteq X \} \)

**Fact:** If \( X \) is convex and nonempty, then \( \text{rint}(X) \) is always non-empty.

The proof is left as an exercise.

**Proposition 2.2** Let \( X \) be a nonempty convex set. Then

\begin{enumerate}
  \item \( \text{int}(X), \text{cl}(X), \text{rint}(X) \) are convex
  \item If \( x_0 \in \text{rint}(X), x \in \text{cl}(X) \), then \( \lambda x_0 + (1 - \lambda)x \in \text{rint}(X), \forall \lambda \in (0, 1] \)
  \item \( \text{cl}(\text{rint}(X)) = \text{cl}(X) \)
  \item \( \text{rint}(\text{cl}(X)) = \text{rint}(X) \)
\end{enumerate}

**Proof:** a) is straightforward, b), c) can be derived as an analogy to the case when \( \text{int}(X) \) is non-empty. d) is due to the following fact: \( \forall x_0 \in \text{rint}(X), x \in \text{rint}(\text{cl}(X)) \), there exist \( y \in \text{cl}(X) \), s.t. \( x \in (x_0, y) \). From b), this implies \( x \in \text{rint}(X) \). \( \blacksquare \)

**Remark** A convex set is perfectly well approximated by its relative interior or closure.

### 2.2 Representation Theorem

**Theorem 2.3** (Caratheodory) Let \( X \subseteq \mathbb{R}^n \) be non empty and \( \dim(X) = d \leq n \). Every point \( x \in \text{conv}(X) \) is a convex combination of at most \( (d+1) \) points, i.e.

\[ \text{Conv}(X) = \{ \sum_{i=1}^{d+1} \lambda_i x_i : x_i \in X, \lambda_i \geq 0, \sum_{i=1}^{d+1} \lambda_i = 1 \} \]

**Proof:** Suppose the minimal representation of \( x \in \text{Conv}(X) \) has \( m \geq d+1 \) terms

\[ x = \sum_{i=1}^{m} \alpha_i x_i, \text{ where } \alpha_i \geq 0, \sum_{i=1}^{m} \alpha_i = 1 \]

The system of linear equations

\[ \begin{cases} 
\sum_{i=1}^{m} \delta_i x_i = 0 \\
\sum_{i=1}^{m} \delta_i = 0 
\end{cases} \]

has non trivial solution.

We can write \( x = \sum_{i=1}^{m} (\alpha_i - t \delta_i) x_i \). Let \( \lambda_i(t) = (\alpha_i - t \delta_i), i = 1, \ldots, m \), we have \( \sum \lambda_i(t) = 1 \).

Let \( t_* = \min \left\{ \frac{\alpha_i}{\delta_i}, \delta_i > 0 \right\} := \frac{\alpha_j}{\delta_j} \), then \( \lambda_i(t_*) > 0, \forall i \neq j \) and \( \lambda_j(t_*) = 0 \). This leads to a smaller representation of \( x \), contradiction! \( \blacksquare \)

**Example** Suppose there are 100 different kinds of herbal tea, everyone of them is a blend of 25 herbs. Donald wants a particular mixture of all herbal teas with equal proportions. What’s the least number of teas he should buy? 26
2.3 Radon and Helley Theorem

**Theorem 2.4 (Radon)** Let $S$ be a collection of $N$ points in $\mathbb{R}^n$ with $N \geq n + 2$. Then we can write $S = S_1 \cup S_2$ s.t. $S_1 \cap S_2 = \emptyset$, and $\text{Conv}(S_1) \cap \text{Conv}(S_2) \neq \emptyset$.

**Proof:** Let $S = \{x_1, ..., x_N\}$ Consider the linear system

$$\begin{cases}
\sum_{i=1}^{N} \gamma_i x_i = 0 \\
\sum_{i=1}^{N} \gamma_i = 0
\end{cases} \implies \text{(n+1) equations, but } N \geq (n + 2) \text{ unknowns}$$

There exists a non-zero solution $\gamma_1, ..., \gamma_N$.

Let $I = \{i : \gamma_i \geq 0\}$, $J = \{j : \gamma_j < 0\}$ and $a = \sum_{i \in I} \gamma_i = -\sum_{j \in J} \gamma_j$, then

$$\sum_{i \in I} \gamma_i x_i = \sum_{j \in J} (-\gamma_j) x_j \implies \sum_{i \in I} \frac{\gamma_i}{a} x_i = \sum_{j \in J} \frac{-\gamma_j}{a} x_j$$

The partition $S_1 = \{x_i, i \in I\}$ and $S_2 = \{x_j : j \in J\}$ gives the desired result. $\blacksquare$

**Theorem 2.5 (Helley)** Let $S_1, ..., S_N$ be a collection of convex sets in $\mathbb{R}^n$. Assume every $(n+1)$ sets of them have a point in common, then all the sets have a point in common.

**Proof:** Let’s prove this by induction on $N$

Base case: $N = n + 1$, obviously true.

Induction step: Assume that the collection of $N(\geq n + 1)$ sets have common point if every $(n + 1)$ of them have common point. We want to show that this holds true for a collection of $N + 1$ sets.

From the assumption, there exists $x_i \in S_1 \cap ... \cap S_{i-1} \cap S_{i+1} \cap ... \cap S_{N+1} \neq \emptyset$. Hence, we obtain $(N + 1) \geq (n + 2)$ points $\{x_1, ..., x_{N+1}\}$.

By Radon’s theorem, we can split $\{x_1, x_2, ..., x_{N+1}\}$ into disjoint sets, whose convex hulls have nonempty intersection. Without loss of generality, let’s assume the two disjoint sets are $\{x_1, \ldots, x_k\}$ and $\{x_{k+1}, \ldots, x_N\}$, and

$$\text{Conv}(\{x_1, ..., x_k\}) \cap \text{Conv}(\{x_{k+1}, ..., x_{N+1}\}) \neq \emptyset.$$ 

Let $z \in \text{Conv}(\{x_1, ..., x_k\}) \cap \text{Conv}(\{x_{k+1}, ..., x_{N+1}\})$. Since $\{x_1, ..., x_k\} \subseteq S_{k+1} \cap ... \cap S_{N+1} \implies z \in \text{Conv}(\{x_1, ..., x_{k+1}\}) \subseteq S_{k+1} \cap ... \cap S_{N+1}$ Since $\{x_{k+1}, ..., x_{N+1}\} \subseteq S_1 \cap ... \cap S_k \implies z \in \text{Conv}(\{x_{k+1}, ..., x_{N+1}\}) \subseteq S_1 \cap ... \cap S_k$. Therefore, $z \in S_1 \cap ... \cap S_{N+1}$. $\blacksquare$

**Remark**

- The theorem is not true for infinite collection: e.g. $S_i = [i, \infty), \cup_{i=1}^{\infty} S_i = \emptyset$
- The theorem is not true if reduce to $(n + 1)$ sets to $n$ sets.

**Corollary 2.6 (Helley)** Let $F$ be any collection of compact convex sets in $\mathbb{R}^n$. If every $(n+1)$ sets have common point, then all sets have $n$ points in common.
Remark Helley’s theorem have many applications, especially for uniform approximation.

Example Consider the optimization problem

$$p_* = \min_{x \in \mathbb{R}^{10}} g_0(x), \quad \text{s.t. } g_i(x) \leq 0, i = 1, ..., 521$$

Suppose \( \forall t \in \mathbb{R}, X_0 = \{x \in \mathbb{R}^{10} : g_0(x) \leq t \} \) is convex, \( X_i = \{x \in \mathbb{R}^{10} : g_i(x) \leq 0 \} \) is convex.

How many constraints can you drop without affecting the optimal value?

**Answer:** You can drop as many as 521 - 11 = 510 constraints. i.e. you just need to keep 11 constraints.

**Proof:** Suppose every 11 constraint relaxation will change the optimal value. \( \forall \{i_1, i_2, ..., i_n\} \subseteq 1, ..., 521, \)

$$\min_{x \in \mathbb{R}^{10}} \{g_0(x) : g_{i_1}(x) \leq 0, ..., g_{i_n}(x) \leq 0\} = p(i_1, ..., i_N) < p_*$$

Since there are only finite combinations, let \( p_{max} \) be the largest among \( p(i_1, ..., i_N), p_{max} < p_* \).

Consider the collection of sets, \( i = 1, ..., 521 \)

$$S_i = \{x \in \mathbb{R}^{10} : g_0(x) \leq p_{max}, g_i(x) \leq 0\}$$

Hence, we have

(i) \( S_i \) is nonempty and convex, and

(ii) every 11 sets of them have non empty intersection

By Helley’s theorem, \( S_1 \cap ... \cap S_{521} \neq \emptyset \) i.e. \( \exists x \in \mathbb{R}^{10} \) s.t. \( g_0(x) < p_* \) and \( g_i(x) \leq 0, \forall i \) Contradiction!

\[ \square \]
In this lecture, we cover the following topics

- Separation Theorems
- The Farkas Lemma
- Duality of Linear Programs

Reference: Boyd & Vandenberghe, Chapter 2.5; Ben-Tal & Nemirovski, Chapter 1.2

### 3.1 Separation of Convex Sets

**Definition 3.1** Let $S$ and $T$ be two nonempty convex sets in $\mathbb{R}^n$, a hyperplane $H = \{ x \in \mathbb{R}^n : a^T x = b \}$ with $a \neq 0$ is said to separate $S$ and $T$ if

- $S \subset H^- = \{ x \in \mathbb{R}^n : a^T x \leq b \}$ and $T \subset H^+ = \{ x \in \mathbb{R}^n : a^T x \geq b \}$
- $S \cup T \not\subset H$

Note that a) implies that

$$\sup_{x \in S} a^T x \leq \inf_{x \in T} a^T x$$

and b) implies that

$$\inf_{x \in S} a^T x < \sup_{x \in T} a^T x$$

The separation is strict if $S \subset \{ x \in \mathbb{R}^n : a^T x \leq b' \}$ and $T \subset \{ x \in \mathbb{R}^n : a^T x \geq b'' \}$, with $b' < b''$. Note that strict separation is equivalent to

$$\sup_{x \in S} a^T x < \inf_{x \in T} a^T x$$

**Question:** When can $S$ and $T$ be separated? strictly separated? Necessary conditions?

**Theorem 3.2** Let $S$ and $T$ be two nonempty convex sets. Then $S$ and $T$ can be separated if and only if $\text{rint}(S) \cap \text{rint}(T) = \emptyset$
Corollary 3.3 Let $S$ be a nonempty convex set and $x_0 \in \partial S$. Then there exists a supporting hyperplane \( H = \{ x : a^T x = a^T x_0 \} \) such that $S \subset \{ x : a^T x \leq a^T x_0 \}$ and $x_0 \in H$.

We will prove a special case of the theorem and corollary.

Theorem 3.4 Let $S$ be closed and convex and $x_0 \not\in S$, Then there exists a hyperplane that strictly separated $x_0$ and $S$.

Proof: Define the projection of $x_0$, denoted as $\text{proj}(x_0)$ to be the point in $S$ that is closest to $x_0$:

\[
\text{proj}(x_0) = \arg \min_{x \in S} \| x - x_0 \|^2_2
\]

Note that $\text{proj}(x_0)$ exists and is unique.

- Existence: due to closedness of $S$
- Uniqueness: If $x_1, x_2$ are both closest to $x_0$ in $S$, then $\| x_1 - x_0 \|^2_2 = \| x_2 - x_0 \|^2_2 = d$. Consider $z = \frac{x_1 + x_2}{2} \in S$, then $\| z - x_0 \| \geq d$. Since $\| (x_0 - x_1) + (x_0 - x_2) \|^2_2 + \| (x_0 - x_1) - (x_0 - x_2) \|^2_2 = 2 \| x_0 - x_1 \|^2_2 + 2 \| x_0 - x_2 \|^2_2$. We have $4 \| x_0 - z \|^2_2 + \| x_1 - x_2 \|^2_2 = 4d^2$. Hence $\| x_1 - x_2 \|^2_2 = 0$, i.e. $x_1 = x_2$.

Next we show that strict separation is given by $H := \{ x : a^T x = b \}$ with $a = x_0 - \text{proj}(x_0)$. $b = a^T x_0 - \frac{\| a \|^2_2}{2}$, i.e. $a^T x < b, \forall x \in S, a^T x_0 > b$.

By definition of projection and convexity, $\forall \lambda \in [0, 1], x \in S$,

\[
\lambda x + (1 - \lambda)\text{proj}(x_0) \in S
\]

Let $\phi(\lambda) = \| \lambda x + (1 - \lambda)\text{proj}(x_0) - x_0 \|^2_2 = \| \text{proj}(x_0) - x_0 + \lambda(x - \text{proj}(x_0)) \|^2_2$

Then

\[
\phi(\lambda) \geq \phi(0), \forall \lambda \in [0, 1]
\]

Hence, $\phi'(0) \geq 0$, i.e. $-2a^T(x - \text{proj}(x_0)) \geq 0$. This implies

\[
a^T x \leq a^T \text{proj}(x_0) = a^T (x_0 - a) = a^T x_0 - \| a \|^2 < a^T x_0 - \frac{\| a \|^2_2}{2} = b
\]

\[\blacksquare\]

Corollary 3.5 Let $S$ and $T$ be two nonempty convex sets and $S \cap T = \emptyset$. Assume $S - T$ is closed, then $S$ and $T$ can be strictly separated.

Proof: Let $Y = S - T$. Since $Y$ is a weighted sum of two convex sets, $Y$ is nonempty and convex. Since $S \cap T, 0 \not\in Y$, from the precious theorem, $\exists a, b$ such that $a^T y < b < 0$. This implies that

\[
a^T x < b + a^T z, \quad \forall x \in S, z \in T
\]

Hence, $\sup_{x \in S} a^T x < \inf_{z \in T} a^T z$, i.e. $S$ and $T$ can be strictly separated. \[\blacksquare\]
3.2 Theorems of alternatives

Theorem 3.6 (Farkas’ Lemma) Exactly one of the following sets must be empty:

(i) \( \{ x \in \mathbb{R}^n : Ax = b, x \geq 0 \} \)

(ii) \( \{ y \in \mathbb{R}^m : A^T y \leq 0, b^T y > 0 \} \)

where \( A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \).

Remark

- System (i) and (ii) are often called strong alternative, i.e. exactly one of them must be feasible.
- Farka’s Lemma is particularly useful to prove infeasibility of a linear program
- Geometric interpretation: let \( A = [a_1 | a_2 | ... | a_n] \),

\[
\text{Cone} \{a_1, ..., a_n\} = \left\{ \sum_{i=1}^{n} x_i a_i : x_i \geq 0, i = 1, ..., n \right\}
\]

(ii) empty \( \iff b \notin \text{Cone} \{a_1, ..., a_n\} \iff \exists y, y^T a_i \leq 0, \forall i = 1, ..., n, y^T b > 0 \)

Farkas’ lemma can be regarded as a special case of the separation theorem.

Proof: First, we show that if system (ii) feasible, then system (i) infeasible. Otherwise, \( 0 < b^T y = (Ax)^T y = x^T (A^T y) \leq 0 \), contradiction!

Second, we show that if system (i) infeasible, then system (ii) feasible. Let \( C = \text{Cone} \{a_1, ..., a_n\} \), then \( C \) is convex and closed. Now that \( b \notin C \), by the separation theorem, \( b \) and \( C \) can be (strictly) separated, i.e.

\[ \exists y \in \mathbb{R}^m, \gamma \in \mathbb{R}, y \neq 0, \text{ such that } y^T z \leq \gamma, \forall z \in C, y^T b > \gamma \]

Since \( 0 \in C \), we have \( \gamma \geq 0 \). Suppose \( \gamma > 0 \), and \( \exists z_0 \in C \) such that \( y^T z_0 > 0 \), then we have \( y^T (\alpha z_0) > \gamma \) when \( \alpha \) is large enough. Hence, it suffices to set \( \gamma = 0 \). Since \( a_1, ..., a_n \in C \), we have \( y^T a_i \leq 0, \forall i = 1, ..., m, \) i.e. \( A^T y \leq 0 \).

Remark: The fact that \( \text{Cone} \{a_1, ..., a_m\} \) is closed is crucial. Note that in general, when \( S \) is not a finite set, \( \text{Cone}(S) \) is not always closed. e.g. the conic hull of a solid circle \( S = \{(x_1, x_2) : x_1^2 + (x_2 - 1)^2 < 1 \} \) is the open halfspace \( \{(x_1, x_2) : x_2 > 0 \} \).
Variant of Farkas’ Lemma  Exactly one of the following two sets must be empty:

1. \( \{ x \in \mathbb{R}^n : Ax \leq b \} \)
2. \( \{ y \geq 0 : A^T y = 0, b^T y < 0 \} \)

Proof: Exercise in HW1.

3.3 LP strong duality

Consider the primal and dual pair of linear programs

\[
\begin{align*}
\text{(P)} \quad & \min & c^T x \\
& \text{s.t.} & Ax = b \\
& & x \geq 0 \\
\text{(D)} \quad & \max & b^T y \\
& \text{s.t.} & A^T y \leq c
\end{align*}
\]

**Theorem 3.7** If \((P)\) has a finite optimal value, then so does \((D)\) and the two values equal each other.

Proof: Exercise in HW 1.

Remark The theorem of alternatives can be generalized to systems with convex constraints, and the strong duality of linear program can be extended to general convex programs.
In this lecture, we cover the following topics

- Convex Functions
- Examples
- Convexity-preserving Operations

Reference: Boyd & Vandenberghe, Chapter 3.1-3.2

4.1 Convex Function

Let $f$ be a function from $\mathbb{R}^n$ to $\mathbb{R}$. The domain of $f$ is defined as $dom(f) = \{x \in \mathbb{R}^n : |f(x)| < \infty\}$. For example,

- $f(x) = \frac{1}{x}$, $dom(f) = \mathbb{R} \setminus \{0\}$
- $f(x) = \sum_{i=1}^{n} x_i \ln(x_i)$, $dom(f) = \mathbb{R}_{++}^n = \{x : x_i > 0, \forall i = 1, ..., n\}$

Definition 4.1 (Convex function) A function $f(x) : \mathbb{R}^n \to \mathbb{R}$ is convex if

(i) $dom(f) \subseteq \mathbb{R}^n$ is a convex set;

(ii) $\forall x, y \in dom(f)$ and $\lambda \in [0, 1]$, $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$.

Geometrically, the line segment between $(x, f(x))$, $(y, f(y))$ sits above the graph of $f$.

Definitions

- A function is called strictly convex if (ii) holds with strict sign, i.e. $f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y)$.
- A function is called $\alpha$-strongly convex if $f(x) - \frac{\alpha}{2} \|x\|^2$ is convex.
- A function is called concave if $-f(x)$ is convex.

Note that strongly convex $\implies$ strictly convex $\implies$ convex
4.2 Examples

1. Simple univariate functions:
   - Even powers: \( x^p, \ p \) is even
   - Exponential: \( e^{ax}, \forall a \in \mathbb{R} \)
   - Negative logarithmic: \(- \log x\)
   - Absolute value: \(|x|\)
   - Negative entropy: \(x \log(x)\)

2. Affine functions: \( f(x) = a^T x + b \)
   - both convex & concave, but not strictly convex/concave

3. Some quadratic functions: \( f(x) = \frac{1}{2} x^T Q x + b^T x + c \)
   - convex if and only if \( Q \succeq 0 \) is positive semi-definite
   - strictly convex if and only if \( Q \succ 0 \) is positive definite
   - special case: \( f(x) = \|Ax - b\|_2^2 \) is convex

4. Norms: A function \( \pi(\cdot) \) is called a norm if
   (a) \( \pi(x) \geq 0, \forall x \) and \( \pi(x) = 0 \) iff \( x = 0 \)
   (b) \( \pi(\alpha x) = |\alpha| \cdot \pi(x), \forall \alpha \in \mathbb{R} \)
   (c) \( \pi(x + y) \leq \pi(x) + \pi(y) \)

Note that norms are convex: \( \forall \lambda \in [0,1], \pi(\lambda x + (1-\lambda)y) \leq \pi(\lambda x) + \pi((1-\lambda)y) = \lambda \pi(x) + (1-\lambda)\pi(y) \) where the inequality comes from (c) and the equality comes from (b).

Examples of norms include:
   - \( l_p \)-norm on \( \mathbb{R}^n \): \( \|x\|_p := (\sum_{i=1}^{n} |x_i|^p)^{1/p}, \) where \( p \geq 1 \)
   - \( Q \)-norm on \( \mathbb{R}^n \): \( \|x\|_Q := \sqrt{x^T Q x}, \) where \( Q > 0 \) is positive definite
• Frobenius norm on $\mathbb{R}^{m \times n}$: $\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |A_{i,j}|^2\right)^{1/2}$

• spectral norm on $\mathbb{S}^n$: $\|A\| = \max_{i=1,\ldots,n} |\lambda_i(A)|$, where $\lambda_i$'s are the eigenvalues of $A$.

5. **Indicator function** $I_c(x) = \begin{cases} 0, & x \in C \\ \infty, & x \not\in C \end{cases}$

The indicator function $I_C(x)$ is convex if the set $C$ is a convex set.

6. **Supporting function**: $I^*_C(x) = \sup_{y \in C} x^T y$

   The support function $I^*_C(x)$ is always convex for any set $C$.

   **Proof:** Note that $\sup_{y \in C} f(y) + g(y) \leq \sup_{y \in C} f(y) + \sup_{y \in C} g(y)$

   Then $\forall x_1, x_2, \lambda \in [0, 1]$

   $$I^*_C(\lambda x_1 + (1 - \lambda)x_2) = \sup_{y \in C} \lambda x_1^T y + (1 - \lambda)x_2^T y$$

   $$\leq \sup_{y \in C} \lambda x_1^T y + \sup_{y \in C} (1 - \lambda)x_2^T y$$

   $$= \lambda I^*_C(x_1) + (1 - \lambda)I^*_C(x_2)$$

7. **More examples**

   • Piecewise linear functions: $\max(a_1^T x + b_1, \ldots, a_k^T x + b_k)$
   • Log of exponential sums: $\log(\sum_{i=1}^{k} e^{a_i^T x + b_i})$
   • Negative log of determinant: $-\log(\det(X))$

   How to show convexity of these functions?

### 4.3 Convexity-Preserving Operators

1. **Taking conic combination:** If $f_i(x), i \in I$ are convex functions and $\alpha_i \geq 0, \forall i \in I$, then

   $$g(x) = \sum_{i \in I} \alpha_i f_i(x)$$

   is a convex function.

   **Proof:** The domain of function $g$

   $$\text{dom}(g) = \cap_{i: \alpha_i > 0} \text{dom}(f_i)$$

   is convex. For any $x, y \in \text{dom}(g), \lambda \in [0, 1]$

   $$g(\lambda x + (1 - \lambda)y) = \sum \alpha_i f_i(\lambda x + (1 - \lambda)y)$$

   $$\leq \sum \alpha_i [\lambda f_i(x) + (1 - \lambda)f_i(y)]$$

   $$= \lambda \sum \alpha_i f_i(x) + (1 - \lambda) \sum \alpha_i f_i(y)$$

   $$= \lambda g(x) + (1 - \lambda)g(y)$$
Remark The property extends to infinite sums and integrals. If $f(x, \omega)$ is convex in $x$ for any $\omega \in \Omega$ and $\alpha(\omega) \geq 0, \forall \omega \in \Omega$, then

$$g(x) = \int_{\Omega} \alpha(\omega) f(x, \omega) d\omega$$

is convex if well defined.

For example if $\eta = \eta(\omega)$ is a well-defined random variable on $\Omega$, and $f(x, \eta(\omega))$ is convex, $\forall \omega \in \Omega$, then $E_{\eta}[f(x, \eta)]$ is a convex function.

2. Taking affine composition If $f(x): \mathbb{R}^n \to \mathbb{R}$ is convex and $A(y): y \mapsto Ay + b$ is an affine mapping from $\mathbb{R}^m$ to $\mathbb{R}^n$, then

$$g(y) := f(Ay + b)$$

is convex on $\mathbb{R}^m$.

Proof: $dom(g) = \{y : Ay + b \in dom(f)\}$ is convex.

$$\forall y_1, y_2 \in dom(g) : g(\lambda y_1 + (1-\lambda)y_2) = f(\lambda(Ay_1 + b) + (1-\lambda)(Ay_2 + b))$$

$$\leq \lambda f(Ay_1 + b) + (1-\lambda)f(Ay_2 + b)$$

$$= \lambda g(y_1) + (1-\lambda)g(y_2)$$

Remark The property extends to the pointwise supremum over an infinite set. If $f(x, \omega)$ is convex in $x$, for $\omega \in \Omega$, then

$$g(x) := \sup_{\omega \in \Omega} f(x, \omega)$$

is convex.

For example, the following functions are convex:
(a) piecewise linear functions: \( f(x) = \max(a_1^T x + b_1, \ldots, a_k^T x + b_k) \)
(b) support function: \( I^*_C(x) = \sup_{y \in C} x^T y \)
(c) maximum distance to any set \( C \): \( d_{\max}(x, C) = \max_{y \in C} \| y - x \|_2 \)
(d) maximum eigenvalue of a symmetric matrix: \( \lambda_{\max}(X) = \max_{\|y\|_2 = 1} y^T X y \)

Indeed, almost every convex function can be expressed as the pointwise supremum of a family of affine functions!

4. Taking convex monotone composition:

- **scalar case** If \( f \) is a convex function on \( \mathbb{R}^n \) and \( F(\cdot) \) is a convex and non-decreasing function on \( \mathbb{R} \), then \( g(x) = F(f(x)) \) is convex.

- **vector case** If \( f_i(x), i = 1, \ldots, m \) are convex on \( \mathbb{R}^n \) and \( F(y_1, \ldots, y_m) \) is convex and non-decreasing (component-wise) in each argument, then

\[
g(x) = F(f_1(x), \ldots, f_m(x))
\]

is convex.

**Proof:** By convexity of \( f_i \), we have

\[
f_i(\lambda x + (1 - \lambda) y) \leq \lambda f_i(x) + (1 - \lambda) f_i(y), \forall i, \forall \lambda \in [0, 1].
\]

Hence, we have for any \( x, y \in \text{dom}(g), \lambda \in [0, 1] \),

\[
g(\lambda x + (1 - \lambda) y) = F(f_1(\lambda x + (1 - \lambda) y), \ldots, f_m(\lambda x + (1 - \lambda) y))
\]
\[
\leq F(\lambda f_1(x) + (1 - \lambda) f_1(y), \ldots, \lambda f_m(x) + (1 - \lambda) f_m(y)) \quad \text{(by monotonicity of \( F \))}
\]
\[
\leq \lambda F(f_1(x), \ldots, f_m(x)) + (1 - \lambda) F(f_1(x), \ldots, f_m(x)) \quad \text{(by convexity of \( F \))}
\]
\[
= \lambda g(x) + (1 - \lambda) g(y) \quad \text{(by definition of \( g \))}
\]

**Remark** Taking pointwise maximum is a special case of the above rule, by setting \( F(y_1, \ldots, y_m) = \max(y_1, \ldots, y_m) \),

\[
\max_{i=1,\ldots,m} f_i(x) = F(f_1(x), \ldots, f_m(x))
\]

is convex.

For example:

- (a) \( e^{f(x)} \) is convex if \( f \) is convex
- (b) \( -\log f(x) \) is convex if \( f \) is concave
- (c) \( \log(\sum_{i=1}^k e^{f_i}) \) is convex if \( f_i \) are convex.

5. Taking Partial minimization: If \( f(x, y) \) is convex in \((x, y) \in \mathbb{R}^n \) and \( Y \) is a convex set, then

\[
g(x) = \inf_{y \in Y} f(x, y)
\]
is convex.

Proof: $\text{dom}(g) = \{x : (x, y) \in \text{dom}(f) \text{ and } y \in C\}$ is a projection of $\text{dom}(f)$, hence is convex. Given any $x_1, x_2$, by definition, for any $\epsilon > 0$, $\exists y_1 \in Y, y_2 \in Y$ s.t.

$$f(x_1, y_1) \leq g(x_1) + \epsilon/2$$
$$f(x_2, y_2) \leq g(x_2) + \epsilon/2$$

For any $\lambda \in [0, 1]$, adding the two equations, we have

$$\lambda f(x_1, y_1) + (1 - \lambda) f(x_2, y_2) \leq \lambda g(x_1) + (1 - \lambda) g(x_2) + \epsilon.$$  

By convexity of $f(x, y)$, this implies

$$f(\lambda x_1 + (1 - \lambda) x_2, \lambda y_1 + (1 - \lambda) y_2) \leq \lambda g(x_1) + (1 - \lambda) g(x_2) + \epsilon.$$  

Hence for any $\epsilon > 0$, $g(\lambda x_1 + (1 - \lambda) x_2) \leq \lambda g(x_1) + (1 - \lambda) g(x_2) + \epsilon$. Letting $\epsilon \to 0$ leads to the convexity of $g$.

Examples

(a) Minimum distance to a convex set: $d(x, C) = \min_{y \in C} \| x - y \|_2$ where $C$ is convex;

(b) Define

$$g(x) = \inf_y \{ h(y) | Ay = x \}$$

is convex if $h$ is convex. This is because $g(x) = \inf_y f(x, y)$, where

$$f(x, y) := \begin{cases} h(x) & Ay = x \\& \\infty & o.w. \end{cases}$$

is convex in $(x, y)$. 

In this lecture, we cover the following topics

- Characterization of Convex Function
  - Epigraph
  - Level set
  - One dimensional property
  - First order condition
  - Second order condition

- Continuity of Convex Functions

References: Boyd & Vandenberghe Chapter 2

5.1 Recall

A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if it satisfies

(i) convexity of domain: $dom(f) = \{x : |f(x)| < +\infty\}$ is convex

(ii) basic convex inequality: if $x, y \in dom(f), \lambda \in [0, 1]$ \[ f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \]

Remark (Extended-value function) $f$ can be extended to a function from $\mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ by setting $f(x) = +\infty$, if $x \not\in dom(f)$. Now (ii) can rewritten as $\forall x, y \in \mathbb{R}^n, \lambda \in [0, 1], f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$

Remark (General convex inequality) $\forall \lambda_i \geq 0, \sum_{i=1}^{m} \lambda_i = 1$, we have by induction that

$$f\left(\sum_{i=1}^{m} \lambda_i x_i\right) \leq \sum_{i=1}^{m} \lambda_i f(x_i)$$

This is also known as Jensen’s inequality. Let $\psi$ be a finite discrete random variable. Then we always have

$$f(\mathbb{E}[\psi]) \leq \mathbb{E}[f(\psi)]$$
5.2 Characterization of Convex Functions

5.2.1 Epigraph

The epigraph of a function is defined as
\[
\text{epi}(f) = \{(x,t) \in \mathbb{R}^{n+1} : f(x) \leq t\}
\]

**Proposition 5.1** \(f\) is convex on \(\mathbb{R}^n\) if and only if its epigraph is a convex set in \(\mathbb{R}^{n+1}\).

**Proof:**

- \((\Leftarrow\text{ part})\) Firstly, \(\text{dom}(f) = \{x : \exists t, \text{ s.t. } (x,t) \in \text{epi}(f)\}\) is convex. Let \((x_1,t_1),(x_2,t_2) \in \text{epi}(f)\), then \((\lambda x_1 + (1 - \lambda)x_2, \lambda t_1 + (1 - \lambda)t_2) \in \text{epi}(f), \forall \lambda \in [0,1]\). By definition of epigraph, this means \(f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda t_1 + (1 - \lambda)t_2\). Particularly, setting \(t_1 = f(x_1), t_2 = f(x_2)\), we have the basic convex inequality.

- \((\Rightarrow\text{ part})\) On the other hand,

\[
f \text{ convex } \Rightarrow f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2), \forall \lambda \in [0,1]
\]

\[
\Rightarrow f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda t_1 + (1 - \lambda)t_2, \forall (x_1,t_1),(x_2,t_2) \in \text{epi}(f)
\]

\[
\Rightarrow \lambda(x_1,t_1) + (1 - \lambda)(x_2,t_2) \in \text{epi}(f)
\]

\[
\Rightarrow \text{epi}(f)\text{is a convex set}
\]

5.2.2 Level set

For any \(t \in \mathbb{R}\), the level set of \(f\) with level \(t\) is defined as
\[
\text{lev}_t(f) = \{x \in \text{dom}(f) : f(x) \leq t\}.
\]

**Proposition 5.2** If \(f\) is convex, then, every level set is convex.

**Proof:** For all \(x_1,x_2 \in \text{lev}_t(f)\) and \(\lambda \in [0,1]\),
\[
f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2) \leq \lambda t + (1 - \lambda)t = t
\]
i.e. \(\lambda x_1 + (1 - \lambda)x_2 \in \text{lev}_t(f)\)

**Remark:** The reverse is not true. A function is called quasi-convex if its domain and all level sets are convex, this holds if and only if
\[
f(\lambda x + (1 - \lambda)y) \leq \max \{f(x), f(y)\}, \forall \lambda \in [0,1]
\]
5.2.3 One-dimensional Property

Recall that a set is convex if and only if its restriction on any line is convex. Similarly, we have

**Proposition 5.3** $f$ is convex if and only if its restriction on any line is convex. i.e. $\forall x, h \in \mathbb{R}^n, \phi(t) = f(x + th)$ is convex on the axis.

**Proof:** ($\implies$) Firstly, $\text{dom}(\phi) = \{t \in \mathbb{R} : x + th \in \text{dom}(f)\}$ is convex. Also, $\forall t_1, t_2 \in \mathbb{R}, \forall \lambda \in [0, 1]$

$$\phi(\lambda t_1 + (1 - \lambda)t_2) = f(x + (\lambda t_1 + (1 - \lambda)t_2)h)$$

$$= f(\lambda (x + t_1h) + (1 - \lambda)(x + t_2h))$$

$$\leq \lambda f(x + t_1h) + (1 - \lambda)f(x + t_2h)$$

$$= \lambda \phi(t_1) + (1 - \lambda)\phi(t_2)$$

Hence, $\phi(t)$ is convex. Proof for the reverse direction is omitted. \[\blacksquare\]

**Remark:** Checking convexity in $\mathbb{R}$ boils down to check convexity of one-dimensional function on the axis. (See example in HW1)

Recall from basic calculus that for a univariate function $f$ on $(a, b)$,

1. if $f$ is differentiable, $f$ convex $\iff f'$ increasing
2. if $f$ is twice-differentiable, $f$ convex $\iff f'' \geq 0$

Similar first and second order conditions hold for general convex functions.

5.2.4 First-order Condition

**Proposition 5.4** Assume $f$ is differentiable, then $f$ is convex if and only if $\text{dom}(f)$ is convex and $f(y) \geq f(x) + \nabla f(x)^T (y - x), \forall x, y$.

**Proof:** ($\Leftarrow$ part) Let $z = \lambda x + (1 - \lambda)y, \forall \lambda \in [0, 1]$

$$f(x) \geq f(z) + \nabla f(z)^T (x - z)$$

$$f(y) \geq f(z) + \nabla f(z)^T (y - z)$$

Hence, we have

$$\lambda f(x) + (1 - \lambda)f(y) \geq f(z) + \nabla f(z)^T (\lambda x + (1 - \lambda)y - z) = f(z) = f(\lambda x + (1 - \lambda)y).$$

($\Rightarrow$ part) By convexity, we have for all $\lambda \in [0, 1],$

$$f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y)$$

$$\Rightarrow f(x + \lambda(y - x)) \leq \frac{f(x) + f(y) - f(x)}{\lambda} f(x)$$

$$\Rightarrow f(y) \geq f(x) + \frac{f(x + \lambda(y - x)) - f(x)}{\lambda}, \forall \lambda \in [0, 1]$$

$$\Rightarrow f(y) \geq f(x) + \nabla f(x)^T (y - x), \text{ letting } \lambda \to 0$$
Remark: The above proposition implies that we have obtain global underestimate of the entire function based on local information \((f(x), \nabla f(x))\). This is an important feature of convex functions. Moreover, one can see that a convex function can be viewed as the supremum of affine function.

5.2.5 Second-order condition

**Proposition 5.5** Assume \(f\) is twice-differentiable, then \(f\) is convex if and only if \(\text{dom}(f)\) is convex and \(\nabla^2 f(x) \succeq 0, \forall x \in \text{dom}(f)\)

**Proof:** (⇐ part) \(\forall x, h, \phi(t) = f(x + th)\) is convex on the axis
\[ \phi''(t) = h^T \nabla^2 f(x + th) h \geq 0 \]
particularly, \(\phi''(0) = h^T \nabla^2 f(x) h \geq 0, \forall h\) Hence \(\nabla^2 f(x) \succeq 0\).

(⇒ part) Any one dimensional restriction
\[ \phi(t) = f(x + th) \text{ is convex since } \phi''(t) \geq 0 \]
Hence \(f\) is convex.

**Example:** \(f(x) = \frac{1}{2} x^T Q x + b^T x + c\) is convex if and only if \(Q \succeq 0\).

5.3 Continuity of Convex Functions

Convex functions are almost everywhere continuous.

**Theorem 5.6** Let \(f : \mathbb{R}^n \rightarrow \mathbb{R}\) be convex, then \(f\) is continuous on \(\text{rint}(\text{dom}(f))\).

**Remark:** Note that \(f\) needs not to be continuous on \(\mathbb{R}^n\) or \(\text{dom}(f)\). e.g.
\[
 f(x) = \begin{cases} 
 1, & x = 0 \\
 0, & x > 0 \\
 +\infty, & \text{o.w.} 
\end{cases}
\]

**Proof:** Without loss of generality, let’s assume \(\text{dim}(\text{dom}(f)) = n, 0 \in \text{int}(\text{dom}(f))\) and \(\{x : \|x\|_2 \leq 1\} \subseteq \text{dom}(f)\). Let us consider the continuity at point 0. Let \(\{x_n\} \rightarrow 0\) with \(\|x_n\|_2 \leq 1\).

(a) \(\lim \sup_{n \rightarrow \infty} f(x_n) \leq f(0)\)

This is because \(x_k = (1 - \|x_k\|_2) \cdot 0 + \|x_k\|_2 \cdot y_k\), where \(y_k = \frac{x_k}{\|x_k\|_2} \in \text{dom}(f)\). By convexity of \(f\), we have
\[
 f(x_k) \leq (1 - \|x_k\|_2) \cdot f(0) + \|x_k\|_2 \cdot f(y_k)
\]
Therefore, \(\lim \sup_{k \rightarrow \infty} f(x_k) \leq f(0)\).
(b) \( \liminf_{n \to \infty} f(x_n) \geq f(0) \)

This is because 0 = \( \frac{1}{\|x_k\|_2 + 1} x_k + \frac{\|x_k\|_2}{\|x_k\|_2 + 1} z_k \), where \( z_k = -\frac{x_k}{\|x_k\|_2} \in \text{dom}(f) \). By the convexity of \( f \), we have

\[
f(0) \leq \frac{1}{\|x_k\|_2 + 1} f(x_k) + \frac{\|x_k\|_2}{\|x_k\|_2 + 1} f(z_k)
\]

Therefore, \( \liminf_{k \to \infty} f(x_k) \geq f(0) \).
In this lecture, we cover the following topics

- subgradient and subdifferential
  - definition and examples, existence, subdifferential properties, calculus

**References**: Bertsekas, Nedich & Ozdaglar, 2003, Chapter 4

### 6.1 Motivation

Convex functions are “essentially” convex sets:

- \( f(x) \) is convex \( \iff \) epi\((f)\) is convex
- \( \sup_{\alpha \in A} f_\alpha(x) \) is convex \( \iff \) \( \cap_{\alpha \in A} \text{epi}(f_\alpha) \) is convex
- "tight affine minorant" of \( f \) \( \iff \) supporting hyperplane of epi\((f)\)

**Question**: Can you find any affine function that underestimates \( f(x) \) and is tight at \( x = 0 \)?

- \( f(x) = \frac{1}{2} x^2 \)
- \( f(x) = |x| \)
- \( f(x) = \begin{cases} -\sqrt{x}, & x \geq 0 \\ +\infty, & \text{o.w.} \end{cases} \)
- \( f(x) = \begin{cases} 1, & x = 0 \\ 0, & x > 0 \\ +\infty, & \text{o.w.} \end{cases} \)
6.2 Subgradient

**Definition 6.1** Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function. A vector $g \in \mathbb{R}^n$ is a subgradient of $f$ at a point $x \in \text{dom}(f)$ if

$$f(y) \geq f(x) + g^T(y - x), \forall y$$

The set of all subgradient at $x$ is called the subdifferential of $f$ at $x$ denoted as $\partial f$. 

**Remark (Subgradient and epigraph)** Note that for any fixed $x \in \text{dom}(f)$

$$g \in \partial f(x) \iff f(y) - g^T y \geq f(x) - g^T x, \forall y$$

$$\iff t - g^T y \geq f(x) - g^T x, \forall (y, t) \in \text{epi}(f)$$

$$\iff \text{The hyperplane } H := \{ (y, t) : (-g, 1)^T(y, t) = (-g, 1)^T(x, f(x)) \}$$

is a supporting plane of $\text{epi}(f)$

**Remark (Differentiable Case)** If $f$ is differentiable at $x \in \text{dom}(f)$, then $\partial f(x) = \{\nabla f(x)\}$ is a singleton.

**Proof:** By definition, let $y = x + \epsilon d, g \in \partial f(x)$, $f(x + \epsilon d) \geq f(x) + \epsilon g^T d$.

$$\frac{f(x + \epsilon d) - f(x)}{\epsilon} \geq g^T d, \forall \epsilon \to 0 \implies \nabla f(x)^T d \geq g^T d, \forall d$$

This only holds when $g = \nabla f(x)$.

**Examples**

1. $f(x) = \frac{1}{2}x^2$, $\partial f(x) = x$

2. $f(x) = |x|$, $\partial f(x) = \begin{cases} \text{sgn}(x), & x \neq 0 \\ [-1,1], & x = 0 \end{cases}$. Note at $x = 0$, $\forall g \in [-1,1], |y| \geq 0 + g \cdot y, \forall y$

3. $f(x) = \begin{cases} -\sqrt{x}, & x \geq 0 \\ +\infty, & \text{o.w.} \end{cases}$, $\partial f(0) = \emptyset$. Note at $x = 0$, $\exists \tilde{g} \in \mathbb{R}$, s.t. $\sqrt{y} \geq 0 + g \cdot y, \forall y \geq 0$

4. $f(x) = \begin{cases} 1, & x = 0 \\ 0, & x > 0 \end{cases}$, $\partial f(0) = \emptyset$. Note at $x = 0$, $\exists \tilde{g} \in \mathbb{R}$, s.t. $\exists 1 + g \cdot y, \forall y \geq 0$

**Theorem 6.2** Let $f$ be a convex function and $x \in \text{dom}(f)$. Then

1. $\partial f(x)$ is convex and closed
2. $\partial f(x)$ is nonempty and bounded if $x \in \text{rint}(\text{dom}(f))$

Proof:

1. Convexity and closedness are evident due to
\[
\partial f(x) = \{ g \in \mathbb{R}^n : f(y) \geq f(x) + g^T(y-x), \forall y \}
\]
\[
= \bigcap_y \{ g \in \mathbb{R}^n f(y) \geq f(x) + g^T(y-x) \}
\]
is the solution to an infinite system of linear inequalities. The intersection of arbitrary number of closed and convex sets is still closed and convex.

2. (Non-empty) W.l.o.g., let's assume $\text{dom}(f)$ is full-dimensional and $x \in \text{int}(\text{dom}(f))$. Since $\text{epi}(f)$ is convex and $(x, f(x))$ belongs to its boundary, by separation theorem, $\exists \alpha = (s, \beta) \neq 0$, s.t.
\[
s^T y + \beta t \geq s^T x + \beta f(x), \forall (y, t) \in \text{epi}(f)
\]
Clearly, we must have $\beta \geq 0$. Since $x \in \text{int}(\text{dom}(f))$, we cannot have $\beta = 0$, Otherwise, to ensure $s^T y \geq s^T x, \forall y \in B(x, y), s = 0$, which is impossible.
Hence, $\beta > 0$, setting $g = -\beta^{-1} s$
\[
f(y) \geq f(x) + g^T(y-x), \forall y
\]
(Bounded) Suppose $\partial f(x)$ is unbounded, i.e. $\exists g_k \in \partial f(x)$, s.t. $\| g_k \|_2 \to \infty$, as $k \to \infty$. Since $x \in \text{int}(\text{dom}(f))$, $\exists \delta > 0$, s.t. $B(x, \delta) \subseteq \text{dom}(f)$. Hence, $y_k = x + \delta \frac{g_k}{\| g_k \|_2} \in \text{dom}(f)$. By convexity,
\[
f(y_k) \geq f(x) + g_k^T(y_k - x) = f(x) + \delta \| g_k \|_2 \to \infty.
\]
However, this contradicts with the continuity of $f$ over $\text{int}(\text{dom}(f))$.

Remark: On the contrary, if $\forall x \in \text{dom}(f), \partial f(x)$ is non-empty, $\text{dom}(f)$ is convex, then $f$ is convex. This is because $\forall x, y \in \text{dom}(f)$ and $\lambda \in [0, 1]$. $z = \lambda x + (1 - \lambda) y \in \text{dom}(f)$. So $\partial f(z) \neq \emptyset$. Let $g \in \partial f(z)$, we have
\[
\begin{align*}
f(x) & \geq f(z) + g^T(x - z) \\
f(y) & \geq f(z) + g^T(y - z)
\end{align*}
\]
\[
\Rightarrow \lambda f(x) + (1 - \lambda) f(y) \geq f(\lambda x + (1 - \lambda) y)
\]

Remark: (Continuity of Subdifferential) Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and continuous. Suppose $x_k \to x$ and $g_k \in \partial f(x_k), g_k \to g$, then $g \in \partial f(x)$.

6.3 Subdifferential and Directional Derivative

Recall that the directional derivative of a function $f$ at $x$ along direction $d$ is
\[
f'(x; d) = \lim_{\delta \to 0^+} \frac{f(x + \delta d) - f(x)}{\delta}
\]
If $f$ is differentiable, then $f'(x; d) = \nabla f(x)^T d$
Lemma 6.3 When \( f \) is convex, the ratio \( \phi(\delta) = \frac{f(x + \delta d) - f(x)}{\delta} \) is non-decreasing of \( \delta > 0 \).

The proof is left as a self exercise.

Theorem 6.4 Let \( f \) be convex and \( x \in \text{int}(\text{dom}(f)) \), then

\[
f'(x; d) = \max_{g \in \partial f(x)} g^T d
\]

Proof: By definition, we have \( f(z + \delta d) - f(x) \geq \delta g^T d \) for all \( \delta \) and \( g \in \partial f(x) \). Hence, \( f'(x; d) \geq g^T d, \forall g \in \partial f(x) \). Moreover, this implies that

\[
f'(x; d) \geq \max_{g \in \partial f(x)} g^T d.
\]

It suffices to show that \( \exists \tilde{g} \in \partial f(x) \), s.t. \( f'(x; d) \leq \tilde{g}^T d \). Consider the two sets

\[
C_1 = \{(y, t) : f(y) < t\}
\]
\[
C_2 = \{(y, t) : y = x + \alpha d, t = f(x) + \alpha f'(x; d), \alpha \geq 0\}
\]

Claim: \( C_1 \cap C_2 = \emptyset \) and \( C_1, C_2 \) are closed and nonempty. This is because \( f(x + \alpha d) \geq f(x) + \alpha f'(x; d), \forall \alpha \geq 0 \). (Due to Lemma 6.3).

By separation theorem, \( \exists (g_0, \beta) \neq 0 \), s.t.

\[
g_0^T (x + \alpha d) + \beta (f(x) + \alpha f'(x; d)) \leq y_0^T y + \beta t, \forall \alpha \geq 0, \forall t > f(y)
\]

One can easily show that \( \beta > 0 \). Let \( \tilde{g} = \beta^{-1} g_0 \),

\[
\tilde{g}^T (x + \alpha d) + f(x) + \alpha f'(x; d) \leq \tilde{g}^T y + f(y), \forall \alpha \geq 0
\]

Let \( \alpha = 1 \) and \( y = x \), we have

\[
\tilde{g} + f(x) \leq \tilde{g}^T y + f(y) \iff -\tilde{g} \in \partial f(x)
\]

Let \( \alpha = 1 \) and \( y = x \), we have

\[
\tilde{g}^T d + f'(x; d) \leq 0 \iff f'(x; d) \leq -\tilde{g}^T d
\]

Therefore, we have shown that \( f'(x; d) = \max_{g \in \partial f(x)} g^T d \)

6.4 Calculus of Subgradient

1. Nonnegative summation: If \( h(x) = \beta_1 f_1(x) + \beta_2 f_2(x) \), with \( \beta_1, \beta_2 \geq 0 \), then

\[
\partial h(x) = \beta_1 \partial f_1(x) + \beta_2 \partial f_2(x)
\]

2. Affine transformation: If \( h(x) = f(Ax + b) \), then

\[
\partial h(x) = A^T \partial h(Ax + b)
\]

3. Pointwise maximum: If \( h(x) = \max_{i \in I} f_i(x), I = \{1, 2, \ldots, m\} \)

\[
\partial h(x) = \text{Conv} \{\partial f_i(x) | i \in I(x)\}, \text{ where } I(x) = \{i \in I : f_i(x) = h(x)\}\]
In this lecture, we cover the following topics

- Closed convex function
- Convex conjugate function
- Conjugate theory
- Introduction to convex programs

References: Bertsekas, Nedich & Ozdaglar, Chapter 7 and Boyd & Vandenberghe Chapter 3.3

7.1 Closed Convex Functions

Recall for a closed convex set \( X \), \( \forall x \not\in X \), \( \exists \) a hyperplane strictly separates \( x \) and \( X \), i.e. \( \exists (a_x, b_x) \), s.t. \( X \subset H_x := \{ y : a_x^T y \leq b_x \} \), and \( x \not\in H_x \). Hence

\[
X = \bigcap_{x \not\in X} \{ y : A_x^T y \leq b_x \}
\]

Any closed convex set is the intersection of closed half-spaces. Intuitively, when translated to convex functions, any convex function with closed and nonempty epigraph is the upper bound of its affine minorants.

Definition 7.1 A function is closed if its epigraph is a closed set.

Remark: Any continuous function is closed, but a closed function is not necessarily continuous. For example,

\[
f(x) = \begin{cases} 
1, & x > 0 \\
0, & x = 0 
+\infty, & o.w.
\end{cases}
\]

Proposition 7.2 Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\} \). Then the following are equivalent.

(i) \( f \) is closed;
(ii) every level set is closed;

(iii) $f$ is lower semi-continuous, i.e. $\forall x \in \mathbb{R}^n, \forall \{x_k\} \rightarrow x, f(x) \leq \liminf_{k \rightarrow \infty} f(x_k)$.

Proof: Self Exercise

Remark Note that a convex function needs not to be closed, e.g.

$$f(x) = \begin{cases} 1, & x = 0 \\ 0, & x > 0 \\ +\infty, & o.w \end{cases}$$

Suppose $f$ is a closed convex function, for a given slope $y \in \mathbb{R}^n$, when is an affine function $y^T x - \beta$ an affine minorant of $f$?

$$f(x) \geq y^T x - \beta \Rightarrow \beta \geq y^T x - f(x), \forall x$$

$$\Rightarrow \beta \geq \sup_{x \in \mathbb{R}^n} \left\{ y^T x - f(x) \right\}$$

The function $f^*(y)$, is known as the conjugate function of $f$.

### 7.2 Convex Conjugate Function

**Definition 7.3** Let $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$: The function

$$f^*(y) = \sup_{x \in \mathbb{R}^n} \{ y^T x - f(x) \} = \sup_{x \in \text{dom}(f)} \{ y^T x - f(x) \}$$

is called the conjugate of function $f$. Also known as Legendre-Fenchel transformation.

**Remark:**

1. (Fenchel’s inequality) $f(x) + f^*(y) \geq x^T y, \forall x, y$
2. $f^*$ is always convex and closed
3. The conjugate of the conjugate function $f^*(y)$, also called biconjugate

$$f^{**}(x) = \sup_{y \in \mathbb{R}^n} \{ x^T y - f^*(y) \}$$

is also closed and convex.

**Theorem 7.4** We have
(a) \( f(x) \geq f^**(x) \)

(b) If \( f \) is closed and convex, then \( f^* = f \).

**Proof:**

(a) By definition of \( f^* \), \( f^*(y) \geq y^T x - f(x), \forall y \). This implies that \( f(x) \geq y^T x - f^*(y), \forall y \). Hence, \( f(x) \geq \sup_y \{ y^T x - f^*(y) \} = f^**(x) \).

(b) Suppose \( f^** \neq f \), then \( \text{epi}(f) \nsubseteq \text{epi}(f^**) \)

\[ \exists (x_0, f^**(x_0)), \text{ s.t. } (x_0, t_0) \in \text{epi}(f^**) \text{ and } (x_0, t_0) \notin \text{epi}(f) \]

Since \( f \) is convex and closed, then \( \text{epi}(f) \) is a closed and convex set. By separation theorem, \( \exists (y, \beta) \neq 0 \text{ s.t. } H = \{(x, t) : y^T x + \beta t = \beta_0 \} \) separates \( \text{epi}(f) \) and \( (x_0, f^**(x_0)) \), Note \( \beta \neq 0 \). w.l.o.g. let \( \beta = -1 \), and

\[ y^T x - t > y^T x_0 - f^*(x_0), \forall (x, t) \in \text{epi}(f) \]

This implies that \( y^T x_0 - f(x_0) > y^T x_0 - f^**(x_0) \). Hence, \( f^**(x_0) > f(x_0) \). Contradiction!

**Examples:**

1. \( f(x) = ax - b \quad f^*(y) = \begin{cases} b, & x = a \\ +\infty, & \text{o.w.} \end{cases} \)

2. \( f(x) = |x| \quad f^*(y) = \begin{cases} 0, & |y| < 1 \\ +\infty, & \text{o.w.} \end{cases} \)

3. \( f(x) = \frac{c}{2} x^2 (c > 0) \quad f^*(y) = \frac{1}{2c} y^2 \)

### 7.3 Convex Optimization

The standard form of an optimization is

\[
\begin{align*}
\min & \quad f(x) \\
\text{s.t.} & \quad g_i(x) \leq 0, i = 1, \ldots, m \\
& \quad h_j(x) = 0, j = 1, \ldots, k
\end{align*}
\]

**Definition 7.5** An optimization problem \( (P) \) is convex if

1. the objective function \( f \) is convex.
2. the inequality constraint functions \( g_1, ..., g_m \) are convex.

3. there is either no equality constraint or only linear equality constraint.

Note the domain of the problem \( D = \text{dom}(f) \cap \text{dom}(g_i) \) is convex.

**Definition 7.6** (Feasibility) The set \( C = \{ x \in \text{dom}(f) : g_i(x) \leq 0, i = 1, ..., m, h_j(x) = 0, j = 1, ..., k \} \) is called the feasible set. Any point \( x \in C \) is called a feasible solution. We say \( (P) \) is feasible if \( C \neq \emptyset \).

**Definition 7.7** (Optimality) The value \( p^* = \inf \{ f(x) : g_i(x) \leq 0, i = 1, ..., m, h_j(x) = 0, j = 1, ..., k \} \) is called the optimal value of \( (P) \). If \( (P) \) is infeasible, we set \( p^* = +\infty \).

- We say \( (P) \) is unbounded below is \( p^* = -\infty \).
- We say \( (P) \) is solvable if \( p^* \) is finite and exists a feasible solution \( x^* \), such that \( p^* = f(x^*) \). We call such \( x^* \), an optimal solution. The set of all optimal solution is called the optimal set.

**Epigraph form of the problem** The standard problem \( (P) \) is equivalent as the problem

\[
\begin{align*}
\min_{x \in \mathbb{R}^n, t \in \mathbb{R}} & \quad t \\
\text{s.t.} & \quad f(x) - t \leq 0 \quad (P^\prime) \\
& \quad g_i(x) \leq 0, i = 1, ..., m \\
& \quad h_j(x) = 0, j = 1, ..., k
\end{align*}
\]

Note that

1. \( (P^\prime) \) is still convex if \( (P) \) is convex

2. \( (x^*, t^*) \) is optimal to \( (P^\prime) \) if and only if \( x^* \) is optimal to \( (P) \) and \( t^* = f(x^*) \)

**Example**

\[
\begin{align*}
\min_{x} & \quad \max_{j=1,...,m} (a_1^T x + b_1, ..., a_m^T x + b_m) \\
\text{s.t.} & \quad Cx = d
\end{align*}
\]

is equivalent to

\[
\begin{align*}
\min_{x,t} & \quad t \\
\text{s.t.} & \quad a_i^T x + b_i - t \leq 0, i = 1, ..., m \\
& \quad Cx = d
\end{align*}
\]
In this lecture, we cover the following topics

- Basics of Convex Programs
- Convex Theorem on Alternatives
- Lagrange Duality

References: Ben-Tal & Nemirovski, Chapter 3.1-3.3

### 8.1 Basics of Convex Programs

The standard form of an optimization problem

$$\min_x f(x)$$

s.t. $g_i(x) \leq 0, i = 1, ..., m$ \ (P)

$h_j(x) = 0, j = 1, ..., k$

The optimal value of (P) is

$$p^* = \begin{cases} 
+\infty, & \text{if no feasible solution} \\
\inf_{x:g_i(x)\leq0,\forall i,h_j(x)=0 \forall j} f(x), & \text{if exists feasible solution.} 
\end{cases}$$

- (P) is **infeasible**, if $p^* = +\infty$
- (P) is **unbounded below**, if $p^* = -\infty$
- (P) is **solvable**, if $\exists$ a feasible solution $x^*$, s.t. $p^* = f(x^*)$
- (P) is **unattainable**, if $|p^*| < \infty$ but $\not\exists$ feasible $x^*$, s.t. $p^* = f(x^*)$. For example, $\min_{x\in(0,1)} e^x$

Given a solution $x^*$,

- $x^*$ is a **global optimum** for (P) if $x^*$ is feasible and $f(x^*) \leq f(x), \forall x$ feasible
• $x^\ast$ is a local optimum for $(P)$ if $x^\ast$ is feasible and $\exists r > 0$, s.t. $f(x^\ast) \leq f(x), \forall$ feasible $x \in B(x^\ast, r)$

**Proposition 8.1** For convex programs, a local optimum is a global optimum.

**Proof:** Let $C$ denote the feasible set, and $x^\ast$ be local optimum. We want to show that $\forall x \in C, f(x^\ast) \leq f(x)$. Let $z = \epsilon x^\ast + (1 - \epsilon) x$. Then $z \in C \cap B(x^\ast, r)$ when $\epsilon$ is small enough. Hence,

$$f(x^\ast) \leq f(z) \leq \epsilon f(x^\ast) + (1 - \epsilon) f(x)$$

i.e., $(1 - \epsilon)f(x^\ast) \leq (1 - \epsilon) f(x)$. Hence, $f(x^\ast) \leq f(x), \forall x \in C$. ■

**Question:** How to verify whether a solution $x^\ast$ is optimal?

In the linear program case, we have shown strong duality between $(P)$ & $(D)$

$$\min \ c^T x$$

$$(P) \quad \text{s.t.} \quad A x = b$$

$$x \geq 0$$

$$\max \ b^T y$$

$$(D) \quad \text{s.t.} \quad A^T y \leq c$$

We know that

$x^\ast$ is optimal $\iff$ $Ax^\ast = b, x \geq 0$ (primal feasibility )

$\exists y^\ast, A^T y^\ast \leq c$  (dual feasibility)

$c^T x^\ast = b^T y^\ast$ (zero-duality gap)

The strong duality is based on the Farkas’ Lemma (or separation theorem). As an analogy, we can derive duality and optimality condition for general convex programs.

### 8.2 Convex Theorems on Alternatives

Consider the general form of convex program

$$\min_{x \in X} f(x)$$

$$\text{s.t.} \quad g_i(x) \leq 0, i = 1, \ldots, m \quad (P)$$

where $X$ is a convex set, $f, g_1, \ldots, g_m$ are convex functions.

**Theorem 8.2** Assume $g_1, \ldots, g_m$ satisfy the Slater condition: $\exists \bar{x} \in X, \text{s.t.} \ g_i(\bar{x}) < 0$. Then exactly one of the following two systems must be empty.
(I) \( \{ x \in X : f(x) < 0, g_i(x) \leq 0, i = 1, ..., m \} \)

(II) \( \{ \lambda \in \mathbb{R}^m : \lambda \geq 0, \inf_{x \in X} \{ f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) \} \geq 0 \} \)

Proof:

1. We first show that system (II) feasible \( \Rightarrow \) system (I) infeasible.

   Suppose (I) is also feasible, \( \exists x_0 \), s.t. \( f(x_0) < 0, g_i(x_0) \leq 0 \). Then

   \[ \forall \lambda \geq 0, \inf_{x \in X} \left\{ f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) \right\} < 0 \]

   Contradiction!

2. We then show that system (I) infeasible \( \Rightarrow \) system (II) feasible.

   Consider the following two sets:
   
   \[ S = \{ u = (u_0, u_1, ..., u_m) \in \mathbb{R}^{m+1} : \exists x \in X, f(x) \leq u_0, g_1(x) \leq u_1, ..., g_m(x) \leq u_m \} \]
   \[ T = \{ u = (u_0, u_1, ..., u_m) \in \mathbb{R}^{m+1} : u_0 < 0, u_1 \leq 0, ..., u_m \leq 0 \} \]

   Note that \( S \) is convex (why?) and nonempty, \( T \) is convex and nonempty and \( S \cap T = \emptyset \).

   By separation theorem, \( \exists a = (a_0, a_1, ..., a_m) \in \mathbb{R}^{m+1} \) and \( a \neq 0 \), s.t.

   \[ \sup_{u \in T} a^T u \leq \inf_{u \in S} a^T u \]

   i.e.

   \[ \sup_{u_0 < 0, u_i \leq 0, i = 1, ..., m} \sum_{i=0}^{m} a_i u_i \leq \inf_{x, u_0, ..., u_m : u_0 \geq f(x), u_i \geq g_i(x), \forall i = 1, ..., m} \sum_{i=0}^{m} a_i u_i \]

   Observe \( a \geq 0 \), hence:

   \[ 0 \leq \inf_{x} a_0 f(x) + a_1 g_1(x) + ... + a_m g_m(x) \]

   Note that \( a_0 > 0 \), otherwise: we have \( (a_1, ..., a_m) \geq 0 \) and \( (a_1, ..., a_m) \neq 0 \),

   \[ \inf_{x} \{ a_1 g_1(x) + ... + a_m g_m(x) \} \geq 0 \]

   However, \( \exists x, \) s.t. \( g_i(x) < 0, \forall i = 1, ..., m. \) This implies

   \[ \inf_{x \in X} a_1 g_1(x) + ... + a_m g_m(x) < 0 \]

   Contradiction!

   Hence, setting \( \lambda_i = \frac{a_i}{a_0}, i = 1, ..., m. \) we have

   \[ \lambda \geq 0 \text{ and } \inf_{x} \left\{ f(x) + \sum \lambda_i g_i(x) \right\} \geq 0 \]

   i.e. system (II) is feasible
Remark (relaxed Slater condition): The Slater condition can be relaxed to accommodate for linear equalities: \( \exists x \in \text{rint}(X) \text{ s.t. } g_i(x) < 0 \) for all \( i = \{1, \ldots, m\} \) such that \( g_i(x) \) is not affine.

Note that in the general case,

\[
x^* \text{ is optimal to } (P) \iff \begin{cases}
\{ x \in X : f(x) \leq f(x^*), g_i(x) \leq 0, i = 1, \ldots, m \} \text{ is feasible} \\
\{ x \in X : f(x) < f(x^*), g_i(x) \leq 0, i = 1, \ldots, m \} \text{ is infeasible}
\end{cases}
\]

\[
\begin{cases}
\{ x \in X : f(x) \leq f(x^*), g_i(x) \leq 0, i = 1, \ldots, m \} \text{ is feasible} \\
\{ \lambda \in \mathbb{R}^m : \lambda \geq 0, \inf_{x \in X} \{ f(x) + \sum \lambda_i g_i(x) \} \geq f(x^*) \} \text{ is feasible}
\end{cases}
\]

Observe that the function \( \inf_{x \in X} \{ f(x) + \sum \lambda_i g_i(x) \} \leq f(x^*) \) for any \( \lambda \geq 0 \). Therefore, the optimality of \( x^* \) implies that there must exist \( \lambda^* \geq 0 \) such that \( \inf_{x \in X} \{ f(x) + \sum \lambda_i^* g_i(x) \} = f(x^*) \).

### 8.3 Lagrange Duality

**Definition 8.3** The function \( L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) \) is called the Lagrange function. This induces two related functions:

\[
\bar{L}(\lambda) = \inf_{x \in X} L(x, \lambda) \\
\bar{L}(x) = \sup_{\lambda \geq 0} L(x, \lambda)
\]

The Lagrange dual of the problem

\[
\begin{align*}
\min & \quad f(x) \\
(P) & \quad \text{s.t. } g_i(x) \leq 0, i = 1, \ldots, m \\
& \quad x \in X
\end{align*}
\]

is defined as

\[
\begin{align*}
\max & \quad \bar{L}(\lambda) \\
(D) & \quad \text{s.t. } \lambda \geq 0
\end{align*}
\]

**Theorem 8.4 (Duality Theorem)** Denote \( \text{Opt}(P) \) and \( \text{Opt}(D) \) as the optimal values to \((P)\) and \((D)\), we have

(a) (Weak Duality) \( \forall \lambda \geq 0, \bar{L}(\lambda) \leq \text{Opt}(D) \). Moreover, \( \text{Opt}(D) \leq \text{Opt}(P) \).

(b) (Strong Duality) If \((P)\) is convex and below bounded, and satisfies Slater condition, then \((D)\) is solvable, and

\[
\text{Opt}(D) = \text{Opt}(P)
\]

**Proof:**
(a) If \((P)\) is infeasible, \(\text{Opt}(P) = \infty\), \(\text{Opt}(D) \leq \text{Opt}(P)\) always hold.

If \((P)\) is feasible, let \(x_0\) be feasible solution, i.e. \(g_i(x_0) \leq 0, x_0 \in X\)

\[
\forall \lambda \geq 0, L(\lambda) = \inf_{x \in X} \left\{ f(x) + \sum \lambda_i g_i(x) \right\} \leq f(x_0) + \lambda_i g_i(x_0) \leq f(x_0)
\]

Hence,

\[
\forall \lambda \geq 0, L(\lambda) \leq \inf \left\{ f(x_0) : x_0 \in X, g_i(x_0) \leq 0, i = 1, ..., m \right\} = \text{Opt}(P)
\]

Furthermore, \(\text{Opt}(D) = \sup_{\lambda \geq 0} L(\lambda) \leq \text{Opt}(P)\)

(b) By optimality, we know that \(\{x \in X : f(x) < \text{Opt}(P), g_i(x) \leq 0, i = 1, ..., m\}\) has no solution.

By convex theorem on alternative and Slater condition, the system \(\{\lambda \geq 0 : L(\lambda) \geq \text{Opt}(P)\}\) has a solution. This implies that \(\text{Opt}(D) = \sup_{\lambda \geq 0} L(\lambda) \geq \text{Opt}(P)\). Combining with part (a), we have \(\text{Opt}(P) = \text{Opt}(D)\).

\[\square\]

**Example:** The Slater condition does not hold in the following example.

\[
\begin{align*}
\min_{x \in X} & \quad e^{-x} \\
\text{s.t.} & \quad \frac{x^2}{y} \leq 0
\end{align*}
\]

where \(X = \{(x, y) : y > 0\}\). Note that there exists no solution in \(x \in X\) such that \(x^2/y < 0\).
In this lecture, we cover the following topics

- Illustrations of Lagrange Duality
- Saddle Point Formulation
- Optimality Conditions (KKT conditions)

References: Bental & Nemirovski Chapter 3.2

9.1 Recall

- Convex program:
  \[
  \min_{x \in X} f(x) \\
  \text{s.t. } g_i(x) \leq 0, \quad i = 1, \ldots, m \quad (P)
  \]
  where \(f(x), g_1, \ldots, g_m\) are convex and \(X\) is convex.

- Lagrange function:
  \[
  L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)
  \]
  where \(\lambda = (\lambda_1, \ldots, \lambda_m)\) is called Lagrange multiplier.

- Lagrange dual function:
  \[
  L(\lambda) := \inf_{x \in X} L(x, \lambda) = \inf_{x \in X} \{f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)\}
  \]
  – Note that \(\forall \lambda \geq 0, L(\lambda) \leq \text{Opt}(P)\)

- Lagrange dual program:
  \[
  \max_{\lambda \geq 0} L(\lambda)
  \]
  We show that \(\text{Opt}(D) = \text{Opt}(P)\) when (relaxed) Slater condition holds
9.2 Illustrations of Lagrange Duality

- **Linear Program Duality**

\[
\begin{align*}
\min_{x} & \quad c^T x \\
\text{s.t.} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

\[
\begin{align*}
\max_{y} & \quad b^T y \\
\text{s.t.} & \quad A^T y \leq c
\end{align*}
\]

First, rewrite the original problem as:

\[
\begin{align*}
\min_{x} & \quad c^T x \\
\text{s.t.} & \quad Ax - b \leq 0 \\
& \quad b - Ax \leq 0 \\
& \quad x \geq 0
\end{align*}
\]

Introducing the multipliers \( \lambda = (\lambda_1, \lambda_2) \geq 0 \), the Lagrange function is

\[
L(x, \lambda) = c^T x + \lambda_1^T (Ax - b) + \lambda_2^T (b - Ax) = (c + A^T \lambda_1 - A^T \lambda_2)^T x + b^T (\lambda_2 - \lambda_1)
\]

The Lagrange dual function is

\[
L(\lambda) = \inf_{x \geq 0} (c + A^T \lambda_1 - A^T \lambda_2)^T x + b^T (\lambda_2 - \lambda_1) = \begin{cases} b^T (\lambda_2 - \lambda_1), & c + A^T \lambda_1 - A^T \lambda_2 \geq 0 \\ -\infty, & \text{o.w.} \end{cases}
\]

The Lagrange dual is \( \max_{\lambda \geq 0} L(\lambda) \), which is equivalent to

\[
\max_{\lambda \geq 0} b^T (\lambda_2 - \lambda_1)
\]

\[
c + A^T (\lambda_1 - \lambda_2) \geq 0
\]

Substituting \( y = \lambda_2 - \lambda_1 \), the formulation above is also equivalent to:

\[
\max_{y} \quad b^T y \\
\text{s.t.} \quad A^T y \leq c
\]

- **Quadratic Program Duality**:

\[
\begin{align*}
\min_{x} & \quad \frac{1}{2} x^T Q x + q^T x \\
\text{s.t.} & \quad Ax \leq b \\
& \quad x \geq 0
\end{align*}
\]

\[
\begin{align*}
\max_{y, \lambda} & \quad \frac{1}{2} y^T Q y + b^T \lambda \\
\text{s.t.} & \quad A^T \lambda - Q y = q \\
& \quad \lambda \geq 0
\end{align*}
\]

where \( Q \succ 0 \).
The Lagrange function is 

\[ L(x, \lambda) = \frac{1}{2} x^T Q x + q^T x + \lambda^T (b - Ax) \]

The Lagrange dual function is 

\[ L(\lambda) = \inf_x \{ \frac{1}{2} x^T Q x + (q - A^T x) + b^T \lambda \} \]

The infimum is achieved when 

\[ Qx = A^T \lambda - q, x = Q^{-1}(A^T \lambda - q) \]

\[ L(\lambda) = -\frac{1}{2} (A^T \lambda - q)^T Q^{-1}(A^T \lambda - q) + b^T \lambda \]

The Lagrange dual is 

\[ \max_{\lambda \geq 0} L(\lambda) \iff \max_{\lambda \geq 0} \frac{1}{2} (A^T \lambda - q)^T Q^{-1}(A^T \lambda - q) + b^T \lambda \iff \max_{\lambda \geq 0, y} -\frac{1}{2} y^T Q y + b^T \lambda \]

s.t. 

\[ A^T \lambda - Qy = q \]

\[ \lambda \geq 0 \]

In both cases discussed above, strong duality holds true.

### 9.3 Saddle Point Formulation

Recall 

\[ (P) : \min_{x \in X} \{ f(x) : g_i(x) \geq 0, i = 1, \ldots, m \} = \inf_{x \in X} \bar{L}(x) = \min_{x \in X} \max_{\lambda \geq 0} L(x, \lambda) \]

\[ (D) : \max_{\lambda \geq 0} L(\lambda) = \max_{\lambda \geq 0} \min_{x \in X} L(x, \lambda) \]

**Definition 9.1** (Saddle point) We call \((x^*, \lambda^*)\), where \(x^* \in X, \lambda^* \geq 0\), a saddle point of \(L(x, \lambda)\) if 

\[ L(x, \lambda^*) \geq L(x^*, \lambda^*) \geq L(x^*, \lambda), \forall x \in X, \lambda \geq 0 \]

**Theorem 9.2** \((x^*, \lambda^*)\) is saddle point of \(L(x, \lambda)\) if and only if \(x^*\) is an optimal solution \((P)\), \(\lambda^*\) is an optimal solution to \((D)\) and \(\text{Opt}(P) = \text{Opt}(D)\).

**Proof:**

\((\Rightarrow)\) Assume \((x^*, \lambda^*)\) is a saddle point,

\[ L(x, \lambda^*) \geq L(x^*, \lambda^*) \geq L(x^*, \lambda), \forall x \in X, \lambda \geq 0 \]

\[ \text{Opt}(P) = \inf_{x \in X} \bar{L}(x) \leq \bar{L}(x^*) = \sup_{\lambda \geq 0} L(x^*, \lambda) = L(x^*, \lambda^*) \]

\[ \text{Opt}(D) = \sup_{\lambda \geq 0} L(\lambda) \geq L(\lambda^*) = \inf_{x \in X} L(x, \lambda^*) = L(x^*, \lambda^*) \]

Hence, \(\text{Opt}(P) \leq L(x^*, \lambda^*) \leq \text{Opt}(D)\)

Combined with weak duality, we have \(\text{Opt}(P) = \text{Opt}(D)\). Hence, \(\text{Opt}(P) = \bar{L}(x^*) = L(x^*, \lambda^*) = L(\lambda^*) = \text{Opt}(D)\). Thus, \(x^*\) solves \((P)\), \(\lambda^*\) solves \((D)\), and \(\text{Opt}(P) = \text{Opt}(D)\)
Assume \((x^*, \lambda^*)\) are optimal solutions to \((P)\) and \((D)\), and \(\text{Opt}(P) = \text{Opt}(D)\). By optimality,

\[
\text{Opt}(P) = \bar{L}(x^*) = \sup_{\lambda \geq 0} L(x^*, \lambda) \geq L(x^*, \lambda^*)
\]

\[
\text{Opt}(D) = L(\lambda^*) = \inf_{x \in X} L(x, \lambda^*) \leq L(x^*, \lambda^*)
\]

Since \(\text{Opt}(D) = \text{Opt}(P)\),

\[
\sup_{\lambda \geq 0} L(x^*, \lambda) = L(x^*, \lambda^*) = \inf_{x \in X} L(x^*, \lambda)
\]

i.e. \((x^*, \lambda^*)\) is a saddle point of \(L(x, \lambda)\).

\[\blacksquare\]

**Remark:**

- The above theorem holds true for any saddle function \(L(x, \lambda)\) and its induced primal and dual problems, not limited to the Lagrange function.
- Saddle point always exists for the Lagrange function of a solvable convex program satisfying the Slater condition. More generally, the existence of saddle points is guaranteed for convex-concave saddle functions over convex compact domains (Minimax Theorem).

### 9.4 Optimality Conditions

**Theorem 9.3:** Let \(x^* \in X\)

(i) (sufficient condition) If there exists \(\lambda^* \geq 0\), such that \((x^*, \lambda^*)\) is a saddle point of \(L(x, \lambda)\), then \(x^*\) is an optimal solution to \((P)\).

(ii) (necessary condition) Assume \((P)\) is convex and satisfies the slater condition. If \(x^*\) is an optimal solution to \((P)\) then \(\exists \lambda^* \geq 0\), s.t. \((x^*, \lambda^*)\) is a saddle point of \(L(x, \lambda)\)

**Proof:**

(i) (sufficient part) Follows from previous theorem.

(ii) (necessary part) By strong duality theorem, \(\exists\) optimal dual solution \(\lambda^* \geq 0\), such that \(\text{Opt}(P) = \text{Opt}(D)\). Hence, following from the previous theorem, \((x^*, \lambda^*)\) is a saddle point of \(L(x, \lambda)\).

\[\blacksquare\]

**Remark** Note that the sufficient condition holds for general constrained program, not necessarily convex ones. However, they are far from being necessary and hardly satisfied.

**Definition 9.4 (Normal Cone)** Let \(X \subset \mathbb{R}^n\) and \(x \in X\). The normal cone of \(X\), denoted as \(N_X(x)\), is the set

\[
N_X(x) = \{ h \in \mathbb{R}^n : h^T(y - x) \geq 0, \forall y \in X \}
\]
Note that $N_X(x)$ is a closed convex cone.

Examples:

- $x \in \text{int}(X)$, $N_X(x) = \{0\}$
- $x \in \text{rint}(X)$, $N_X(x) = L^\perp$ where $L$ = Linear subspace parallel to Aff$(X)$
- $X = \{ x : a_i^T x \geq b_i, i = 1, ..., m \}$, $x \notin \text{int}(X)$.

$$N_X(x) = \text{Cone}\{a_i | a_i^T x = b_i\}$$

**Theorem 9.5** Let $(P)$ be a convex program and let $x^*$ be a feasible solution. Assume $f, g_1, ..., g_m$ are differentiable at $x^*$.

(a) (sufficient condition) If there exists $\lambda^* \geq 0$ satisfying

1) $\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) \in N_X(x^*)$

2) $\lambda_i^* g_i(x^*) = 0, \forall i = 1, ..., m$

then $x^*$ is optimal solution to $(P)$ (Karush-Kuhn-Tucker, KKT condition (1951))

(b) (necessary condition) If $(P)$ also satisfies the slater condition. then the above condition is also necessary for $x^*$ to be optimal.

**Proof:**

(a) (sufficient part) Under the KKT condition

1) implies that $L(x, \lambda^*) \geq L(x^*, \lambda^*), \forall x \in X$ because

$$L(x, \lambda^*) \geq L(x^*, \lambda^*) + \nabla_x L(x^*, \lambda^*)(x - x^*)$$

as the last part is non-negative

2) + feasibility of $x^*$ implies $L(x^*, \lambda^*) \geq L(x^*, \lambda), \forall \lambda \geq 0$ because

$$L(x^*, \lambda^*) = f(x^*) + \sum_{i=1}^{m} \lambda_i g_i(x^*) = f(x^*) \geq f(x^*) + \sum \lambda_i g_i(x^*) = L(x^*, \lambda)$$

Hence $(x^*, \lambda^*)$ is a saddle point of $L(x, \lambda)$.

(b) (necessary part) From previous theorem, there exists $\lambda \geq 0$ such that $(x^*, \lambda^*)$ is a saddle point of $L(x, \lambda)$. We have

$$L(x, \lambda^*) \geq L(x^*, \lambda^*), \forall x \in X \Rightarrow (y - x^*)^T \nabla L_x(x^*, \lambda^*) = \lim_{\epsilon \to 0} \frac{L(x^* + \epsilon(y - x^*), \lambda^*) - L(x^*, \lambda^*)}{\epsilon} \geq 0$$

$$\Rightarrow \nabla_x L(x^*, \lambda^*) \in N_X(x^*)$$
L(x^*, \lambda^*) \geq L(x^*, \lambda), \forall \lambda \geq 0 \Rightarrow \sum_{i=1}^{m} \lambda_i^* g_i(x^*) \geq \sum_{i=1}^{m} \lambda_i g_i(x^*), \forall \lambda \geq 0
\Rightarrow \sum_{i=1}^{m} \lambda_i^* g_i(x^*) \geq 0 \quad \text{(note we also have} \lambda_i^* g_i(x^*) \leq 0)\Rightarrow \lambda_i^* g_i(x^*) = 0, \forall i

This leads to the KKT condition.

Remark To summarize, \((x^*, \lambda^*)\) is an optimal primal-dual pair if it satisfies:

- **Primal feasibility:** \(x^* \in X, g_i(x^*) \leq 0\)
- **Dual feasibility:** \(\lambda^* \geq 0\)
- **Lagrange optimality:** \(\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) \in N_X(x^*)\)
- **Complementary slackness:** \(\lambda_i^* g_i(x^*) = 0, \forall i = 1, \ldots, m\)

Example: Given \(a_i > 0, i = 1, \ldots, n\), solve the problem

\[
\min_x \sum_{i=1}^{n} \frac{a_i}{x_i}
\text{ s.t. } x > 0 \quad \sum_{i=1}^{n} x_i \leq 1
\]

The Lagrange function \(L(x, \lambda) = \sum_{i=1}^{n} \frac{a_i}{x_i} + \lambda(\sum_{i=1}^{n} x_i - 1)\). The KKT optimality conditions yield

\[
\begin{cases}
x_i^* > 0, \sum_{i=1}^{n} x_i^* \leq 1 \\
\lambda^* \geq 0 \\
-\frac{a_i}{(x_i^*)^2} + \lambda^* = 0 \\
\lambda^* (\sum_{i=1}^{n} x_i^* - 1) = 0
\end{cases}
\Rightarrow x_i = \sqrt{\frac{a_i}{\lambda^*}} \quad \text{and} \quad \sum_{i=1}^{m} x_i = 1 \Rightarrow \begin{cases}
\lambda^* = \left(\sum_{i=1}^{n} \sqrt{a_i}\right)^2 \\
x_i^* = \frac{\sqrt{a_i}}{\sum_{i=1}^{n} \sqrt{a_i}}, i = 1, \ldots, m
\end{cases}
\]
In this lecture, we cover the following topics

- Saddle Point and Minimax Problems
- Minimax Theorem (Sion-Kakutani Theorem)

References: Bental & Nemirovski Chapter 3.4

### 10.1 Saddle Point and Minimax Problem

Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be two non-empty sets. Let $L(x, y) : X \times Y \to \mathbb{R}$ be a real-valued function. We say $(x^*, y^*)$ is a saddle point of $L(x, y)$ if

$$L(x, y^*) \geq L(x^*, y^*) \geq L(x^*, y), \forall x \in X, y \in Y$$

**Game Theory Interpretation:** This can be viewed as a two player zero-sum game.

- Player 1 chooses $x \in X$
- Player 2 chooses $y \in Y$
- After both players reveal their decisions, player 1 pays to player 2 the amount $L(x, y)$.
- Player 1 is interested in minimizing his loss, while player 2 is interested in maximizing his gain.
- The strategy $(x^*, y^*)$ is called a (Nash) equilibrium if no player has an interactive to change his chosen strategy after considering the opponent’s action.

**Question:** What should players do to optimize their profit?

Considering the following two situations

- **Situation I:** Player 1 choose $x$ first and player 2 choose $y$ knowing the choice of player 2

$$\min_{x \in X} \bar{L}(x) := \max_{y \in Y} L(x, y)$$
Situation II: Player 2 choose $y$ first and player 1 choose $x$ knowing the choice of player 1

$$
\max_{y \in Y} L(y) := \min_{x \in X} L(x, y)
$$

Two Induced Problems:

$$(P) : \quad \min_{x \in X} \max_{y \in Y} L(x, y) = \min_{x \in X} \bar{L}(y)
$$

$$(D) : \quad \max_{y \in Y} \min_{x \in X} L(x, y) = \max_{y \in Y} L(x)
$$

Intuitively, it is better for players to play second. Indeed, there is an inequality:

**Proposition 10.1**

$$
sup_{y \in Y} \inf_{x \in X} L(x, y) \leq \inf_{x \in X} sup_{y \in Y} L(x, y).
$$

**Proof:** This is because

$$
\forall x \in X : \inf_{y \in Y} L(x, y) \leq L(x, y), \forall y
$$

$$
\Rightarrow \forall x \in X : \sup_{y \in Y} \inf_{x \in X} L(x, y) \leq \sup_{y \in Y} L(x, y)
$$

$$
\Rightarrow \sup_{y \in Y} \inf_{x \in X} L(x, y) \leq \inf_{x \in X} \sup_{y \in Y} L(x, y)
$$

On the other hand, if $(x^*, y^*)$ is a saddle point.

$$
L(x, y) \geq L(x^*, y) \geq L(x^*, y^*), \quad \forall x \in X, y \in Y
$$

$$
\Rightarrow \inf_{x \in X} L(x, y^*) \geq L(x^*, y^*) \geq \sup_{y \in Y} L(x^*, y)
$$

$$
\Rightarrow \sup_{y \in Y} \inf_{x \in X} L(x, y) \geq \inf_{x \in X} \sup_{y \in Y} L(x, y)
$$

Hence, if $(x^*, \lambda^*)$ is a saddle point, then

$$
sup_{y \in Y} \inf_{x \in X} L(x, y) = \inf_{x \in X} \sup_{y \in Y} L(x, y)
$$

**Remark:**

- If there exists a saddle point, there is no advantage to the players of knowing the opponent’s choice. Moreover, the equilibrium, minimax and maximin all give the same solution.

- Saddle points do not always exist. For example. $L(x, y) = (x - y)^2, X = [0, 1], Y = [0, 1]$

  $$
  \bar{L}(x) = \sup_{y \in [0,1]} L(x, y) = \max \left\{ x^2, (x - 1)^2 \right\}, \quad \inf_{x \in X} \sup_{y \in Y} L(x, y) = \frac{1}{4}
  $$

  $$
  L(y) = \inf_{x \in [0,1]} L(x, y) = 0, \quad \sup_{y \in Y} \inf_{x \in X} L(x, y) = 0
  $$

- As already shown in last lecture: Saddle point exists if and only the induced problems $(P)$ and $(D)$ are both solvable and the optimal values equal to each other. Moreover, the saddle points are pairs $(x^*, \lambda^*)$, where $x^*$ is optimal to $(P)$ and $\lambda^*$ is optimal to $(D)$. 
10.2 Existence of Saddle Points

Lemma 10.2 (Minimax Lemma) Let \( f_i(x), i = 1, \ldots, m \) be convex and continuous on a convex compact set \( X \). Then
\[
\min_{x \in X} \max_{1 \leq i \leq m} f_i(x) = \min_{x \in X} \sum_{i=1}^{m} \lambda_i^* f_i(x)
\]
for some \( \lambda^* \in \mathbb{R}^m \) such that, \( \lambda_i^* \geq 0, i = 1, \ldots, m \) and \( \sum_{i=1}^{m} \lambda_i^* = 1 \)

Remark: Let \( L(x, \lambda) = \sum_{i=1}^{m} \lambda_i f_i(x), \Delta = \{ \lambda : \lambda \geq 0, \sum_{i=1}^{m} \lambda_i = 1 \} \) be a standard simplex. The above lemma implies: \( L(x, \lambda) \) has a saddle point on \( X \times \Delta \). Since
\[
\max_{\lambda \in \Delta} \min_{x \in X} \sum_{i=1}^{m} \lambda_i f_i(x) \geq \min_{x \in X} \sum_{i=1}^{m} \lambda_i^* f_i(x) = \min_{x \in X} \max_{\lambda \in \Delta} \sum_{i=1}^{m} \lambda_i f_i(x)
\]

Proof: Consider the epigraph form of the problem \( \min_{x \in X} \max_{1 \leq i \leq m} f_i(x) \):
\[
\min_{x, t} \quad t \\
\text{ s.t. } \quad f_i(x) - t \leq 0, i = 1, \ldots, m \\
\quad (x, t) \in X_t
\]
where \( X_t = \{(x, t) : x \in X, t \in \mathbb{R}\} \). The optimal value \( t^* = \min_{x \in X} \max_{1 \leq i \leq m} f_i(x) \).

Note that the above problem satisfies the Slater condition and is solvable (why?)
Hence, there exists \((x^*, t^*) \in X_t \) and \( \lambda^* \geq 0 \), such that \((x^*, t^*; \lambda^*) \) is a saddle point of the Lagrange function:
\[
L(x, t; \lambda) = t + \sum_{i=1}^{m} \lambda_i(f_i(x) - t) = (1 - \sum_{i=1}^{m} \lambda_i)t + \sum_{i=1}^{m} \lambda_i f_i(x)
\]

Therefore:
\[
\begin{align*}
\frac{\partial L}{\partial t} (x^*, t^*; \lambda^*) &= 1 - \sum_{i=1}^{m} \lambda_i^* = 0 \\
\sum_{i=1}^{m} \lambda_i^* (f_i(x^*) - t^*) &= 0
\end{align*}
\Rightarrow \begin{cases} 
\sum_{i=1}^{m} \lambda_i^* = 1 \\
\sum_{i=1}^{m} \lambda_i^* f_i(x^*) = t^*
\end{cases}
\]

This implies that \( \exists \lambda_i^* \geq 0, \sum_{i=1}^{m} \lambda_i^* = 1, \) such that
\[
\min_{x \in X} \max_{1 \leq i \leq m} f_i(x) = t^* = \min_{(x, t) \in X_t} L(x, t; \lambda^*) = \min_{x \in X} \sum_{i=1}^{m} \lambda_i^* f_i(x)
\]

Theorem 10.3 (Minima Theorem, Sion-Kakutani)
Let \( X \) and \( Y \) be two convex compact sets. Let \( L(x, y) : X \times Y \to \mathbb{R} \) be a continuous function that is convex in \( x \in X \) for every fixed \( y \in Y \) and concave in \( y \in Y \) for every fixed \( x \in X \). Then \( L(x, y) \) has a saddle point on \( X \times Y \), and
\[
\min_{x \in X} \max_{y \in Y} L(x, y) = \max_{y \in Y} \min_{x \in X} L(x, y)
\]
Proof: We should prove that

\[(P) : \min_{x \in X} \tilde{L}(x) := \max_{y \in Y} L(x, y)\]

\[(D) : \max_{y \in Y} L(y) := \min_{x \in X} L(x, y)\]

are both solvable with equal optimal values.

1. X and Y are compact, L(x, y) is continuous, both \(\tilde{L}(x)\) and \(L(y)\) are continuous and attain their optimum on compact set. It is sufficient to show \(\text{Opt}(D) \geq \text{Opt}(P)\), i.e.

\[\max_{y \in Y} \min_{x \in X} L(x, y) \geq \min_{x \in X} \max_{y \in Y} L(x, y)\]

Consider the sets \(X(y) = \{x \in X : L(x, y) \leq \text{Opt}(D)\}\)

2. Note that \(X(y)\) is nonempty, compact and convex for any \(y \in Y\). We show that every collection of these sets has a point in common.

Suppose \(\exists y_1, ..., y_m\) s.t. \(X(y_1) \cap ... \cap X(y_m) = \emptyset\). This implies:

\[\min_{x \in X} \max_{i = 1, ..., m} L(x, y_i) > \text{Opt}(D)\]

By Minimax Lemma, \(\exists \lambda^*_i \geq 0\) and \(\sum_{i=1}^{m} \lambda^*_i = 1\), such that

\[\min_{x \in X} \max_{i = 1, ..., m} L(x, y_i) = \min_{x \in X} \sum_{i=1}^{m} \lambda^*_i L(x, y_i)\]

\[\leq \min_{x \in X} \sum_{i=1}^{m} \lambda^*_i y_i\]

\[= L(\bar{y}) \leq \text{Opt}(D)\]

where \(\bar{y} = \sum_{i=1}^{m} \lambda^*_iy_i\) and the first inequality is by concavity of \(L(x, y)\). The result here leads to a contradiction! Hence, every finite collection of \(X(y)\) has a common point.

3. By Helley’s theorem, all of these sets \(\{X(y) : y \in Y\}\) has a common point. Therefore, \(\exists x^* \in X : x^* \in X(y), \forall y \in Y\), which means \(\exists x^* \in X, L(x^*, y) \leq \text{Opt}(D), \forall y \in Y\). Hence \(\text{Opt}(P) \leq \text{Opt}(D)\)
In this lecture, we cover the following topics

- Some Remarks on Minimax Problem
- Some Remarks on Optimality Conditions
- Polynomial Solvability of Convex Programs

11.1 Minimax Problem

Recall in the last lecture, we have discussed

[Minimax Theorem] If \( X \) and \( Y \) are convex compact sets, \( L(x,y), X \times Y \rightarrow \mathbb{R} \) is convex-concave and continuous, then \( L(x,y) \) has a saddle point on \( X \times Y \), and

\[
\min_{x \in X} \max_{y \in Y} L(x,y) = \max_{y \in Y} \min_{x \in X} L(x,y)
\]

**Remark 1:** Based on the proof analysis, we can immediately see that some of the assumptions can be relaxed. Here is a general result:

**Theorem 11.1** Let \( X \) and \( Y \) be convex and one of them is compact. Let \( L(x,y) \) be lower-continuous (l.s.c.) and quasi-convex in \( x \in X \) and upper semi-continuous (u.s.c) and quasi-concave in \( y \in Y \). Then

\[
\min_{x \in X} \max_{y \in Y} L(x,y) = \max_{y \in Y} \min_{x \in X} L(x,y)
\]

**Remark 2:** The compactness is sufficient but not necessary, see for examples:

1. \( \min_x \max_y (x + y) = \infty \neq -\infty = \max_y \min_x (x + y) \)
2. \( \min_x \max_{0 \leq y \leq 1} (x + y) = -\infty = -\infty = \max_{0 \leq y \leq 1} \min_x (x + y) \)
3. \( \min_x \max_{y \leq 1} (x + y) = -\infty = -\infty = \max_{y \leq 1} \min_x (x + y) \)
Remark 3: Minimax problem could also arise from Fenchel dual:
Let $f(x)$ be closed and convex, then $f = f^{**}$. We can equivalently write $f$ as

$$f(x) = \max_{y \in \mathbb{R}^n} \{ y^T x - f^*(x) \}$$

Hence

$$\min_{x \in X} f(x) = \min_{x \in X} \max_{y \in \mathbb{R}^n} \{ y^T x - f^*(x) \}$$

Remark 4: Alternative optimization does not necessarily converge to the saddle point.
Consider the saddle point problem

$$\min_{-1 \leq x \leq 1} \max_{-1 \leq y \leq 1} xy$$

The problem has a unique saddle point: $(0, 0)$. Start with any $(x_0, y_0)$, where $x_0 > 0$ and do alternative maximization over $y$ and minimization over $x$:

$$(x_0, y_0) \Rightarrow (x_0, 1) \Rightarrow (-1, 1) \Rightarrow (-1, -1) \Rightarrow (1, -1) \Rightarrow (1, 1) \Rightarrow ...$$

This will not converge to the saddle point.

11.2 Optimality Conditions for Convex Programs

- General Constrained Differentiable Case:

$$\min_{x \in X} f(x) \quad g_i(x) \leq 0, \quad i = 1, ..., m \quad (P)$$

We have already shown that for a feasible solution $x^*$ to be optimal:

$$x^* \in X \text{ is optimal for } (P) \iff \exists \lambda^* \geq 0 \text{ s.t.} \begin{align*}
(a) \quad & \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) \in N_X(x^*) \\
(b) \quad & \lambda_i^* g_i(x^*) = 0, \forall i = 1, ..., m
\end{align*} \quad (11.1)$$

- Simple Constrained Differentiable Case:

$$\min_{x \in X} f(x)$$

Proposition 11.2 Assume $X$ is convex and $f(x)$ is convex and differentiable at $x^*$. Then

$$x^* \in X \text{ is optimal } \iff \nabla f(x^*) \in N_X(x^*), \text{i.e. } \nabla f(x^*)^T (x - x^*) \geq 0, \forall x \in X$$

Proof:

($\Leftarrow$) $f(x) \geq f(x^*) + \nabla f(x^*)^T (x - x^*) \geq f(x^*), \forall x \in X$

($\Rightarrow$) $\nabla f(x^*) (x - x^*) = \lim_{\epsilon \to 0} \frac{f(x^* + \epsilon(x-x^*)) - f(x^*)}{\epsilon} \geq 0$
Remark

1. If \( x^* \in \text{int}(X) \), then \( x^* \) is optimal \( \Leftrightarrow \nabla f(x^*) = 0 \)
2. (Unconstrained case): If \( X = \mathbb{R}^n \), then \( x^* \) is optimal \( \Leftrightarrow \nabla f(x^*) = 0 \)
3. For general (non-convex) optimization problems, \( \nabla f(x^*) = 0 \) is only a necessary but not sufficient condition for \( x^* \) to be optimal.

Nondifferentiable Case:

**Proposition 11.3** Assume \( f(x) \) is convex on \( X \), and \( x^* \in \text{rint} (\text{dom}(f)) \), then

\[ x^* \in X \text{ is optimal} \Leftrightarrow \exists g \in \partial f(x^*), \text{ s.t. } g^T(x - x^*) \geq 0, \forall x \in X \]

**Proof:**

\[ \begin{align*}
(\Leftarrow) & \quad f(x) \geq f(x^*) + g^T(x - x^*) \geq f(x^*), \forall x \in X \\
(\Rightarrow) & \quad \max_{g \in \partial f(x^*)} g^T(x - x^*) = f'(x^*; x - x^*) = \lim_{\epsilon \to 0} \frac{f(x^* + \epsilon(x - x^*)) - f(x^*)}{\epsilon} \geq 0
\end{align*} \]

Remark In the unconstrained case when \( X = \mathbb{R}^n \), \( x^* \) is optimal \( \Leftrightarrow 0 \in \partial f(x^*) \)

### 11.3 Solve Convex Programs

\[
\begin{align*}
\min & \quad f(x) \\
\text{s.t.} & \quad g_i(x) \leq 0, \quad i = 1, \ldots, m \quad (P) \\
& \quad x \in X
\end{align*}
\]

In practice, to solve a problem means to find a "approximate" solution to (P) with a small inaccuracy \( \epsilon > 0 \).

**Measure of accuracy of an approximate solution \( \hat{x} \):** Some function \( \epsilon(\hat{x}) \) such that \( \epsilon(\hat{x}) \geq 0 \) and \( \epsilon(\hat{x}) \to 0 \) as \( \hat{x} \to x^* \). For instance:

1. \( \epsilon(\hat{x}) = \inf_{x^* \in X_{\text{opt}}} \| \hat{x} - x^* \| \), where \( X_{\text{opt}} \) is the optimal set.
2. \( \epsilon(\hat{x}) = f(\hat{x}) - \text{Opt}(P) \), where \( \text{Opt}(P) \) is the optimal value.
3. \( \epsilon(\hat{x}) = \max \left( f(\hat{x}) - \text{Opt}(P), \max_{1 \leq i \leq m} [g_i(\hat{x})]_+ \right) \), where \( u_+ = \max(u, 0) \)
**Black-box oriented numerical method:** We assume the objective and constraints can be accessed through oracles.

- **zero-order oracle:** \( \mathcal{O} = (f(x), g_1(x), ..., g_m(x)) \)
- **first-order oracle:** \( \mathcal{O} = (\partial f(x), \partial g_1(x), ..., \partial g_m(x)) \)
- **second-order oracle:** \( \mathcal{O} = (\nabla^2 f(x), \nabla^2 g_1(x), ..., \nabla^2 g_m(x)) \)
- **separation oracle:** given \( x \), either reports \( x \in X \) or returns a separator, i.e. a vector \( e \neq 0 \), such that \( e^T x \geq \sup_{y \in X} e^T y \). Note that when \( x \not\in \text{int}(X) \), a separator does exist.

**Complexity of a numerical method** \( M \): Given an input \( \epsilon > 0 \), a problem instance \( P \),

- **oracle complexity**: number of oracles required to solve the problem \( (P) \) up to accuracy \( \epsilon > 0 \)
- **arithmetric complexity**: number of arithmetic operation (bit-wise operation) requirement to solve the problem \( (P) \) up to accuracy \( \epsilon > 0 \)

A solution method \( M \) for a family \( P \) of problems is called **polynomial** if \( \forall p \in P \), the arithmetic complexity

\[
\text{Compl}_M(\epsilon, p) \leq O(1) \left[ \frac{\text{dim}(P)}{\epsilon} \right]^\alpha \cdot \ln(\text{V}(P)/\epsilon)
\]

where \( \text{V}(P) \) is some data-dependent quantity.

We say the family \( P \) of problems **polynomially solvable**, i.e. **computationally tractable**, if it admits polynomial solution methods.

### 11.4 Convex Problems are Polynomially Solvable

**Illustration:** One-dimensional Case

\[
\min_{x \in [a,b]} f(x)
\]

- **Zero-order line search:** Assume there exists a unique minimizer
  - Initialize a localizer \( G_1 = [a, b] \) for \( x^* \)
  - At each iteration, choose \( a_t, b_t \in G_t \), update the localizer
    \[
    G_{t+1} \leftarrow \begin{cases} 
    [a_t, b_t] \cap G_t, & \text{if } f(a_t) \leq f(b_t) \\
    [a_t, b_t] \cap G_t, & \text{if } f(a_t) > f(b_t)
    \end{cases}
    \]

  If we choose \( a_t, b_t \) that split \([a, b]\) into equal length, \(|G_{t+1}| = \frac{2}{3}|G_t|\). We get linear convergence.

- **First-order line search (bisection)**
- Initialize a localizer $G_1 = [-R, R] \supset [a, b]$
- At each iteration, compute the midpoint $c_t$ of $G_t = [a_t, b_t]$
  
  if $c_t \notin [a, b]$, 
  
  $$G_{t+1} = \begin{cases} [a_t, c_t], & \text{if } c_t > b \\ [c_t, b_t], & \text{if } c_t < a \end{cases}$$

  if $c_t \in [a, b]$ and $f'(c_t) \neq 0$
  
  $$G_{t+1} = \begin{cases} [a_t, c_t], & \text{if } f'(c_t) > 0 \\ [c_t, b_t], & \text{if } f'(c_t) < 0 \end{cases}$$

  otherwise, this implies $c_t$ is optimal

Note that $x^* \in G_t$ and $|G_{t+1}| = \frac{1}{2} |G_t|$, we get linear convergence.
In this lecture, we cover the following topics

- Center of Gravity Method
- Ellipsoid Method

References: Bental & Nemirovski, Chapter 7

12.1 Convex Program

We aim to solve the convex optimization problem

$$\min_{x \in X} f(x)$$

where $f$ is convex and $X \subseteq \mathbb{R}^n$ is closed, bounded convex with non-empty interior (also called a convex body).

Let $r, R$ be such that $\{x : \|x - c\|_2 \leq r\} \subseteq X \subseteq \{x : \|x\|_2 \leq R\}$ for some $c$.

Setup: We assume that the objective and constraint set can be accessed through

- Separation Oracle for $X$: a routine that given an input $x$, either reports $x \in X$ or returns a vector $\omega \neq 0$, s.t. $\omega^T x \geq \sup_{y \in X} \omega^T y$
- First Order Oracle for $f$: a routine that given an input $x$, returns a subgradient $g \in \partial f(x)$, i.e. $f(y) \geq f(x) + g^T (y - x), \forall y$.
- Zero Order Oracle for $f$: a routine that given an input $x$, returns the function value $f(x)$.

Example:

$$\min_{x} \max_{1 \leq j \leq J} f_j(x)$$

s.t. $g_i(x) \leq 0, \quad i = 1, \ldots, m$

where $f_j(x), 1 \leq j \leq J$ and $g_i(x), 1 \leq i \leq m$ are convex and differentiable. Here, we have

$$f(x) = \max_{1 \leq j \leq J} f_j(x)$$

$$X = \{x \in \mathbb{R}^n : g_i(x) \leq 0, i = 1, \ldots, m\}$$
1. We can build first order oracle for \( f \) if at a given \( x \), we can compute \( f_1(x), \ldots, f_J(x) \), and \( \nabla f_1(x), \ldots, \nabla f_J(x) \). This is because:

\[
\partial f(x) \supseteq \text{Conv} \{ \partial f_j(x), j \text{ such that } f(x) = f_j(x) \}
\]

2. We can build separation oracle for \( X \) if at given \( x \), we can compute \( g_1(x), \ldots, g_m(x) \) and \( \nabla g_1(x), \ldots, \nabla g_m(x) \). This is because

\[
x \in X \iff g_i(x) \leq 0, \forall i = 1, \ldots, m
\]

and

\[
x \notin X \iff \exists i', \text{ s.t. } g_{i'}(x) > 0
\]

\[
\implies \nabla g_{i'}(x)^T(y - x) \leq g_{i'}(y) - g_{i'}(x) \leq 0, \forall y \in X
\]

\[
\implies \omega^T x \geq \sup_{y \in X} \omega^T y, \text{ for } \omega = \nabla g_{i'}(x)
\]

### 12.2 Center of Gravity Method (Levin, 1965; Newman, 1965)

We consider the simple cutting plane scheme

- Initialize \( G_0 = X \)
- At iteration \( t = 1, 2, \ldots, T \), do
  - Compute the center of gravity: \( x_t = \frac{1}{\text{vol}(G_{t-1})} \int_{x \in G_{t-1}} x \, dx \)
  - Call the first order oracle and obtain \( g_t \in \partial f(x_t) \)
  - Set \( G_t = G_{t-1} \cap \{ y : g_t^T(y - x_t) \leq 0 \} \)
- Output \( \hat{x}_T \in \arg \min_{x \in \{ x_1, \ldots, x_T \}} f(x) \)

**Lemma 12.1** Let \( C \) be a centered convex body in \( \mathbb{R}^n \) with \( \int_{x} x \, dx = 0 \). Then \( \forall \omega \neq 0 \)

\[
\text{Vol}(C \cap \{ x : \omega^T x \leq 0 \}) \leq \left( 1 - \left( \frac{n}{n+1} \right)^n \right) \text{Vol}(C) \leq \left( 1 - \frac{1}{e} \right) \text{Vol}(C)
\]

As an immediate result of the center of gravity method, we have

- \( x^* \in G_t, \forall t \geq 1 \) because \( X_{opt} \subseteq \{ y : f(y) \leq f(x_t) \} \subseteq \{ y : g_t^T(y - x_t) \leq 0 \} \)
- \( \text{Vol}(G_t) \leq (1 - \frac{1}{e})^{T} \text{Vol}(X) \)
\textbf{Theorem 12.2} The approximate solution generated by the center of gravity methods satisfies:

\[ f(\hat{x}_T) - f^* \leq (1 - \frac{1}{e})^\frac{T}{n} \operatorname{Var}_X(f) \]

where \( \operatorname{Var}_X(f) = \max_{x \in X} f(x) - \min_{x \in X} f(x) \), and \( f^* \) is the optimal value.

\textbf{Proof:} Let \( \delta \in ((1 - \frac{1}{e})^\frac{T}{n}, 1) \), and consider the neighborhood of \( x^* \)

\[ X_\delta = \{ x^* + \delta(x - x^*) : x \in X \} \]

\[ \operatorname{Vol}(X_\delta) = \delta^n \operatorname{Vol}(X) > (1 - \frac{1}{e})^T \operatorname{Vol}(X) \geq \operatorname{Vol}(G_T) \]

Hence \( X_\delta / G_T \neq 0 \). Let \( y = x^* + \delta(z - x^*) \in X_\delta / G_T \) for some \( z \in X \).

Thus, for certain \( t^* \leq T \), we have \( y \in G_{t^*-1} / G_{t^*} \). Since \( y \notin G_{t^*} \), we have \( g_{t^*}(y - x_{t^*}) > 0 \), hence \( f(y) > f(x_{t^*}) \). Since \( y = x^* + \delta(z - x^*) \), by convexity of \( f \),

\[ f(y) = f(\delta z + (1 - \delta)x^*) \leq \delta f(z) + (1 - \delta)f(x^*) = f(x^*) + \delta[f(z) - f(x^*)] \]

\[ \leq f(x^*) + \delta \operatorname{Var}_X(f) \]

Hence \( f(\hat{x}_T) \leq f(x_{t^*}) \leq f^* + \delta \operatorname{Var}_X(f) \). Let \( \delta \to (1 - \frac{1}{e})^\frac{T}{n} \), we got the desired result. \( \square \)

\textbf{Remarks:}

1. To obtain a solution with small inaccuracy \( \epsilon > 0 \), the number of oracles needed are polynomially dependent on the dimension.
   - separation oracle: \( N(\epsilon) = n \log \left( \frac{\operatorname{Var}_X(f)}{\epsilon} \right) \)
   - first order oracle: \( N(\epsilon) = n \log \left( \frac{\operatorname{Var}_X(f)}{\epsilon} \right) \)
   - zero order oracle: \( N(\epsilon) = n \log \left( \frac{\operatorname{Var}_X(f)}{\epsilon} \right) \)

2. The rate of convergence of center of gravity methods is exponentially fast and this is usually called a \textit{linear rate}.

3. The center of gravity is not necessarily polynomial and cannot be used as a computational tool because finding the center of gravity at each step can be extremely difficult, even for polytopes.

4. We can use ellipsoid as the localizer, so that is is easy to compute the center.

\textbf{12.3 Ellipsoid Method (Shor, Nemirovsky, Yudin, 1970s)}

\textbf{Ellipsoid.} Let \( Q \) be a symmetric positive definite matrix, and \( c \) be the center, an ellipsoid is uniquely characterized by \( (c, Q) \):

\[ E(c, Q) = \{ x \in \mathbb{R}^n : (x - c)^T Q^{-1} (x - c) \leq 1 \} \]

\[ = \{ x = c + Q^{\frac{1}{2}} u : u^T u \leq 1 \} \]
The Volume of $E(c,Q)$ is

$$\text{Vol}(E(c,Q)) = \text{Det}(Q^{\frac{1}{2}})\text{Vol}(B_n)$$

where $B_n$ is a $n$-dimensional Euclidean ball with $\text{Vol}(B_n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$.

**Proposition 12.3** Let $E = E(c,Q) = \{x \in \mathbb{R}^n : (x - c)^TQ^{-1}(x - c) \leq 1\}$ be an ellipsoid. Let $H_+ = \{x : \omega^T x \leq \omega^T c\}$ be a half space with $\omega \neq 0$ that pass through the center $c$. Then the half ellipsoid $E \cap H_+$ can be contained by the ellipsoid $E^+ = E(c^+, Q^+)$ with

$$c^+ = c - \frac{1}{n+1}q$$

$$Q^+ = \frac{n^2}{n^2 + 1}(Q - \frac{2}{n+1}qq^T)$$

where $q = \frac{Q\omega}{\sqrt{\omega^T Q \omega}}$ is the step from $c$ to the boundary of $E$. Moreover,

$$\text{Vol}(E^+) \leq \exp\left\{-\frac{1}{2n}\right\} \text{Vol}(E)$$

**The Ellipsoid Method.** The algorithm works as follows:

- Initialize $E(c_0,Q_0)$ with $c_0 = 0, Q_0 = R^2I$
- At iteration $t = 1,2,...,T$, do
  - Call separation oracle with the input $c_{t-1}$
  - If $c_{t-1} \notin X$, obtain a separator $u \neq 0$
  - If $c_{t-1} \in X$, call first order oracle and obtain a subgradient $\omega \in \partial f(c_t)$
  - Set the new ellipsoid $E(c_t,Q_t)$ with
    $$c_t = c_{t-1} - \frac{1}{n+1} \frac{Q_{t-1}\omega}{\sqrt{\omega^T Q_{t-1} \omega}}$$
    $$Q_t = \frac{n^2}{n^2 - 1}(Q_{t-1} - \frac{2}{n+1} \frac{Q_{t-1}\omega^T Q_{t-1} \omega}{\omega^T Q_{t-1} \omega})$$
- Output
  $$\hat{x}_T = \min_{c \in \{c_1,\ldots,c_T\} \cap X} f(c)$$

As a immediate result,

$$\text{Vol}(E_t) \leq \exp\left\{-\frac{t}{2n}\right\} \text{Vol}(E_0) = \exp\left\{-\frac{t}{2n}\right\} R^n \text{Vol}(B_n)$$
**Theorem 12.4** The approximate solution generated by the Ellipsoid methods after $T$ steps, for $T > 2n^2 \log\left(\frac{R}{r}\right)$ satisfies:

$$f(\hat{x}_T) - f^* \leq \frac{R}{r} \operatorname{Var}_X(f) \exp\left\{-\frac{T}{2n^2}\right\}$$

*Proof:* Similar as the proof for the center of gravity method. Set $\delta \in \left(\frac{R}{r} \exp\left\{-\frac{T}{2n^2}\right\}, 1\right)$. Then $X_{\delta} = \{x^* + \delta(x - x^*) : x \in X\}$

$$\operatorname{Vol}(X_{\delta}) = \delta^n \operatorname{Vol}(X) \geq \delta^n \gamma^n \operatorname{Vol}(B_n) > R^n \exp\left\{-\frac{T}{2n}\right\} \operatorname{Vol}(B_n) \geq \operatorname{Vol}(E_t)$$

Hence, $X_{\delta}/E_t \neq \emptyset$. The rest of the proof follows similarly as in Theorem 12.2. □

**Remark:** The oracle complexity for Ellipsoid method is $O(n^2 \log(\frac{1}{\epsilon}))$, which is only slightly worse than the center of gravity method.

**Remark:** The Ellipsoid method works for any general convex problems as long as they admit separation and first order oracles. Moreover, the algorithm is polynomial if it takes only polynomial time to call those oracles. For instance, for linear programs with $n$ variables and $m$ constraints, it takes $O(nm)$ computation cost for the separation oracle.

In summary, here are some advantages and disadvantages of Ellipsoid method:

+ universal
+ simple to implement and steady for small size problems
+ low order dependence on the number of functional constraints
− quadratic growth on the size of problem, inefficient for large-scale problems.
In this lecture, we cover the following topics

- Generalized Inequality Constraints and Cones
- Conic Programs: LP, SOCP, SDP

References: Ben-Tal & Nemirovski. *Lectures on Modern Convex Optimization*, Chapter 1.4

### 13.1 Generalized Inequality

In order to extend the linear constraints to nonlinear convex constraints, we may extend the standard component-wise inequality to a generalized vector inequality.

$$Ax \geq b \Rightarrow Ax \succ b$$

where the partial ordering “$\succ$” also satisfies:

- (i) reflexivity: $a \succ a$
- (ii) anti-symmetry: $a \succ b$ and $b \succ a$ implies $a = b$
- (iii) transitivity: $a \succ b$ and $b \succ c$ implies $a \succ c$
- (iv) homogeneity: $a \succ b$ and $\lambda \in \mathbb{R}_+$ implies $\lambda a \succ \lambda b$
- (v) additivity: $a \succ b$ and $c \succ b$ implies $a + c \succ b + d$

**Remark 1:** The generalized inequality on $\mathbb{R}^m$ is completely identified by the set $K = \{a \in \mathbb{R}^m : a \succ 0\}$ via the rule

$$a \succ b \iff a - b \in K$$

This is because:

- $a \succ b$ and $-b \succ -b$ (by (i)) implies $a - b \succ 0$
- $a - b \succ 0$ and $b \succ b$ (by (i)) implies $a \succ b$
Remark 2 The set $K$ satisfies

1. $K$ is nonempty: $0 \in K$
2. $K$ is closed w.r.t addition: $a, b \in K \Rightarrow a + b \in K$
3. $K$ is closed w.r.t multiplication $a \in K, \lambda \in \mathbb{R} \Rightarrow \lambda a \in K$
4. $K$ is pointed: $a, -a \in K \Rightarrow a = 0$

Equivalently, $K$ must be a nonempty pointed convex cone. Indeed, this condition is both necessary and sufficient for the set $K$ to define a partial ordering “$\geq_K$” via the rule:

$$a \geq_K b \iff a - b \in K$$

Remark 3: The standard inequality $a \geq b \iff a - b \in \mathbb{R}^m_+$. In addition, $K = \mathbb{R}^m_+$ is also closed and has non-empty interior.

13.2 Conic Programs

Definition 13.1 Let $K$ be a regular cone (the cone is convex, closed, pointed and with a nonempty interior) and ”$\geq_K$” be the induced inequality. The optimization problem:

$$\min_x \ c^T x$$

s.t. $Ax \geq_K b$ (CP)

is called a conic program associated with the cone $K$.

Examples of Regular Cones

- Non-negative Orthant:
  $$\mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x_i \geq 0, i = 1, ..., m\}$$

- Lorentz Cone (a.k.a second order/ ice-cream cone)
  $$L^m = \left\{ x \in \mathbb{R}^m : x_m \geq \sqrt{\sum_{i=1}^{m-1} x_i^2} \right\}$$

- Semidefinite Cone:
  $$S^m_+ = \{A \in S^m : A \succ 0\}$$
  is the set of $m \times m$ symmetric positive semidefinite matrices.
• Linear Program: $K = \mathbb{R}^m_+ = \mathbb{R} \times \ldots \times \mathbb{R}$ is a direct product of nonnegative orthant:

$$\min_x \{ c^T x : a_i^T x - b_i \geq 0, i = 1, \ldots, m \}$$  \hspace{1cm} (LP)

where the linear map $Ax - b_i := [a_i^T x - b_i; \ldots; a_m^T x - b_m]$

• Conic Quadratic Program (a.k.a Second Order Conic Program): $K = L^{m_1} \times \ldots \times L^{m_k}$

$$\min_x \{ c^T x : \| D_i x - d_i \|_2 \leq e_i^T x - f_i, i = 1, \ldots, m \}$$  \hspace{1cm} (SOCP)

where the linear map $Ax - b := \begin{bmatrix} D_1 x - d_1; e_1^T x - f_1; \ldots; D_k x - d_k; e_k^T x - f_k \end{bmatrix}$

• Semidefinite Program:

$$\min_x \left\{ c^T x : \sum_{i=1}^n x_i A_i - B \succeq 0 \right\}$$  \hspace{1cm} (SDP)

where the linear map $Ax - b = \sum_{i=1}^m x_i A_i - B : \mathbb{R}^n \to \mathbb{S}^m$, where $A_1, \ldots, A_n, B \in \mathbb{S}^m$

Remark: $(LP) \subseteq (SOCP)$
This is because $a_i^T x - b_i \geq 0 \iff \begin{bmatrix} 0 \\ a_i^T x - b_i \end{bmatrix} \in L^2$.

Remark: $(SOCP) \subseteq (SDP)$
This is because:

$$\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \in \mathbb{L}^m \iff \begin{bmatrix} x_m & x_1 & x_2 & \ldots & x_{m-1} \\ x_1 & x_m \\ x_2 & x_m & \ddots \\ \vdots \\ x_{m-1} & \ldots & \ldots & x_m \end{bmatrix} \succeq 0$$

Lemma 13.2 (Schur Complement) Let $S = \begin{bmatrix} P & Q^T \\ Q & R \end{bmatrix}$ be symmetric with $R \succ 0$. Then

$$S \succeq 0 \text{ if and only if } P - Q^T R^{-1} Q \succeq 0$$

Proof:

$$S \succeq 0 \iff \forall u,v : [u^T, v^T] \begin{bmatrix} P & Q^T \\ Q & R \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \succeq 0$$

$$\iff \inf_{u,v} u^T P u + 2v^T Q u + v^T R v \geq 0$$

$$\iff \inf_u u^T (P - Q^T R^{-1} Q) u \geq 0$$

$$\iff P - Q^T R^{-1} Q \succeq 0$$
To show \((\text{SOCP}) \subseteq (\text{SDP})\)

1. If \(x \in L_m^m\) and \(x \neq 0\), then \(x_m > 0\) and
\[
x_m \geq \frac{x_1^2 + \ldots + x_{m-1}^2}{x_m}
\]
The above lemma implies \(Ax \geq 0\). If \(x \in L_m^m\) and \(x = 0\), then \(Ax = 0 \in S_m^+\).

2. If \(Ax \geq 0\) and \(Ax \neq 0\), then \(x_m > 0\). Then by Schur complement Lemma, we have:
\[
x_m - \frac{x_1^2 + \ldots + x_{m-1}^2}{x_m} \geq 0 \Rightarrow x \in L_m^m
\]
If \(Ax = 0\), then \(x = 0 \in L_m^m\)

### 13.3 Examples

1. \(L_2\)-norm minimization

   (a) \(\min_{x \in \mathbb{R}^n} \|x\|_2\)

   Note that
   \[
   \min_{x \in \mathbb{R}^n} \|x\|_2 \iff \min_{x,t} t \iff \min_{x,t} t
   \]
   \[
   \text{s.t. } t \geq \|x\|_2 \quad \text{s.t. } \begin{bmatrix} x \\ t \end{bmatrix} \succeq L_{n+1}^n 0
   \]

   (b) \(\min_{x \in \mathbb{R}^n} x^T x\)

   Note that
   \[
   \min_{x \in \mathbb{R}^n} x^T x \iff \min_{x,t} t \iff \min_{x,t} t
   \]
   \[
   \text{s.t. } t \geq x^T x \quad \text{s.t. } \begin{bmatrix} 2x \\ t \end{bmatrix} \succeq L_{n+2}^n 0
   \]

2. Quadratic problems: \(\min_{x \in \mathbb{R}^n} x^T Q x + q^T x\) where \(Q = LL^T \succeq 0\)

   Note that
   \[
   \min_{x \in \mathbb{R}^n} x^T Q x + q^T x \iff \min_{x,t} t \iff \min_{x,t} t
   \]
   \[
   \text{s.t. } t \geq x^T Q x + q^T x \quad \text{s.t. } \begin{bmatrix} 2L^T x \\ t - q^T x - 1 \\ t - q^T x + 1 \end{bmatrix} \succeq L_{n+2}^n 0
   \]
3. Spectral norm minimization:

Spectral norm: \( \| A \|_2 = \lambda_{\text{max}}(A) \), where \( A \) is symmetric

Note that

\[
\min_{x \in \mathbb{R}^n} \left\| \sum_{i=1}^{n} x_i A_i \right\|_2 \iff \min_{x \in \mathbb{R}^n, t} t \iff \min_{x, t} t
\]

s.t. \( t \geq \lambda_{\text{max}}(\sum_{i=1}^{n} x_i A_i) \)

s.t. \( t I_m - \sum_{i=1}^{n} x_i A_i \succeq 0 \)

where \( A_1, ..., A_n \in S^m \)
In this lecture, we cover the following topics

- Dual Cone
- Conic Duality Theorem

References: Ben-Tal & Nemirovski. *Lectures on Modern Convex Optimization*, Chapter 1.4

### 14.1 Motivation

Recall the LP Duality:

\[
\begin{align*}
\min_{x} & \quad c^T x \\ (P) & \quad \text{s.t. } Ax = b \\ & \quad x \geq 0
\end{align*}
\]

\[
\begin{align*}
\max_{y} & \quad b^T y \\ (D) & \quad \text{s.t. } A^T y \leq c
\end{align*}
\]

\[
\begin{align*}
\min_{x} & \quad c^T x \\ (LP) & \quad \text{s.t. } Ax \geq b
\end{align*}
\]

\[
\begin{align*}
\max_{y} & \quad b^T y \\ (LD) & \quad \text{s.t. } A^T y = c \\ & \quad y \geq 0
\end{align*}
\]

Now consider the conic program

\[
\begin{align*}
\min_{x} & \quad c^T x \\ (CP) & \quad \text{s.t. } Ax \geq_{K} b
\end{align*}
\]

\[
\begin{align*}
\max_{y} & \quad b^T y \\ (CD) & \quad ?
\end{align*}
\]
Observe that

\[ Ax \geq b \Rightarrow \forall y \geq 0, y^T (Ax) \geq y^T b \]
\[ \Rightarrow \forall y \geq 0 \text{ and } A^T y = c, c^T x \geq b^T y \]
\[ \Rightarrow c^T x \geq \max \{ b^T y : y \geq 0, A^T y = c \} \]

Now that \( Ax \geq_K b \Rightarrow y^T (Ax) \geq y^T b \) for which \( y \)?

### 14.2 Dual Cone

**Definition 14.1** Let \( K \) be a nonempty cone. The set

\[ K^* = \{ y : y^T x \geq 0, \forall x \in K \} \]

is called the dual cone to \( K \). Note that \( K^* \) is a closed cone.

**Proposition 14.2** Let \( K \) be a closed cone and \( K^* \) be its dual cone. Then:

1. \( (K^*)_s = K \)
2. \( K \) is pointed iff \( K^* \) has non-empty interior
3. \( K \) is a regular cone iff \( K^* \) is a regular cone

**Proof:** Self-exercise.

**Examples:** We call these self-dual cones.

- \( (\mathbb{R}^m_+)^*_s = \mathbb{R}^m_+ \)
- \( (L^m)^*_s = L^m \)
- \( (S^m_+)^*_s = S^m_+ \)

### 14.3 Conic Duality

The dual of the conic program

\[ (CP) \quad \min_x \{ c^T x : Ax \geq_K b \} \]

is given by

\[ (CD) \quad \max_y \{ A^T y = c, y \geq_K 0 \} \]
**Theorem 14.3** (Weak Conic Duality) The optimal value of $(CD)$ is a lower bound of the optimal value of $(CP)$.

**Theorem 14.4** (Strong Conic Duality) If the primal $(CP)$ is bounded below and strictly feasible, i.e., $\exists x_0$, s.t. $Ax_0 \succ_K b$. Then the dual $(CD)$ is solvable and the optimal values equal to each other.

*Proof:* Let $p^*$ be the optimal value of the primal $(CP)$. It’s sufficient to show that $\exists y^*$ feasible to $(CD)$, such that $b^T y^* \geq p^*$.

Suppose $c = 0$, $p^* = 0$, $\exists y^* = 0$, s.t. $y^* \succeq_K$, $A^T y^* = c$ and $b^T y^* = p^*$.

Now consider $c \neq 0$. Let the set $M = \{Ax - b: c^T x \leq p^*\}$. Note that $M$ is a nonempty set.

**Claim 1:** $M \cap \text{int}(K) = \emptyset$

This is because: Suppose $\exists \bar{x}$, s.t. $A\bar{x} \in \text{int}(K)$ and $c^T \bar{x} \leq p^*$. Then there exists a small enough neighborhood of $\bar{x}$ that are feasible. Since $c \neq 0$, there exists a point $\tilde{x}$ in this neighborhood such that $c^T \tilde{x} < c^T \bar{x} \leq p^*$. This contradicts with the fact that $p^*$ is the optimal value.

By Separation Theorem, $\exists y \neq 0$, s.t.

$$\sup_{z \in M} y^T z \leq \inf_{z \in \text{int}(K)} y^T z$$

Note that $\inf_{z \in \text{int}(K)} y^T z = 0$, since $\text{int}(K)$ contains the ray$\{tz: t \geq 0\}, \forall z \in \text{int}(K)$.

Hence, we have $y \in K^*$ and

$$\sup_{x: c^T x \leq p^*} y^T (Ax - b) \leq 0 \quad (14.1)$$

Therefore, it must hold that $\lambda c = A^T y$ for some $\lambda \geq 0$. Otherwise, the supremum goes to infinity.

**Claim 2:** $\exists \lambda > 0$, s.t. $A^T y = \lambda c$

This is because: Suppose $\lambda = 0$, $A^T y = 0$, (14.1) implies $-b^T y \leq 0$. Since $(CP)$ is strictly feasible, $\exists x_0$, s.t. $Ax_0 - b \in \text{int}(K)$. We already show $y \in K_*$, then $y^T (Ax_0 - b) > 0$, which implies $y^T b > 0$. Contradiction!

Now let $y^* = \frac{y}{\lambda}$, we obtain $y^* \in K_*, A^T y^* = c$ and $c^T x - (y^*)^T b \leq 0, \forall x$ such that $c^T x \leq p^*$.

Therefore, $y^*$ is dual feasible and $b^T y^* \geq p^*$

**Corollary 14.5** If the dual $(CD)$ is bounded above and strictly feasible, i.e., $\exists y$, s.t. $y \succ_K 0$ and $A^T y = c$

then the primal $(CP)$ is solvable and the optimal values equal to each other.

**Corollary 14.6** If both $(CP)$ and $(CD)$ are strictly feasible, the both are solvable with equal optimal values.
Theorem 14.7 (Optimality Conditions) Suppose at least one of (CP) and (CD) is bounded and strictly feasible, then the feasible primal-dual pair \((x^*, y^*)\) is a pair of optimal primal-dual solutions iff

1. (Zero duality gap) if and only if \(c^T x^* - b^T y^* = 0\)

2. (Complementary Slackness) if and only if \((Ax^* - b)^T y^* = 0\)

Proof: Let \(p^*\) and \(d^*\) be the optimal values of (CP) and (CD)

\[
c^T x^* - b^T y^* = (c^T x^* - p^*) + (d^* - b^T y^*) + (p^* - d^*)
\]

All three terms on RHS are \(\geq 0\). And \(c^T x^* - b^T y^* = 0\) iff \(c^T x^* = p^*, d^* = b^T y^*\) and \(p^* = d^*\)

Remark In the case of LP, the strictly feasibility is not required for strong duality and it is not required for a program to be solvable. However, in the genetic case of CP, this is not necessary true.

Example 1: A conic problem can be strictly feasible and bounded, but NOT solvable.

\[
\begin{array}{c}
\min_{x_1, x_2} & x_1 \\
\text{s.t.} & \begin{bmatrix} x_1 - x_2 \\ 1 \\ x_1 + x_2 \end{bmatrix} \geq_{L^3} 0 \\
\end{array} \iff \begin{array}{c}
\min_{x_1, x_2} & x_1 \\
\text{s.t.} & 4x_1x_2 \geq 1 \\
& x_1 + x_2 > 0 \\
\end{array}
\]

Example 2: A conic problem can be not strictly feasible yet solvable, and the dual is infeasible.

\[
\begin{array}{c}
\min_{x_1, x_2} & x_2 \\
\text{s.t.} & \begin{bmatrix} x_1 \\ x_2 \\ x_1 \end{bmatrix} \geq_{L^3} 0 \\
\end{array} \iff \begin{array}{c}
\min_{x_1, x_2} & x_2 \\
\text{s.t.} & x_2 = 0 \\
& x_1 \geq 0 \\
\end{array} \iff \begin{array}{c}
\max_{\lambda} & 0 \\
\text{s.t.} & \begin{bmatrix} \lambda_1 + \lambda_3 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
& \lambda \geq_{L^3} 0 \\
\end{array}
\]
In this lecture, we cover the following topics

- SOCP Duality
- SDP Duality
- SDP Relaxations for Non-Convex Quadratic Programs

References: Ben-Tal & Nemirovski. Lectures on Modern Convex Optimization, Chapter 2.1, 3.1

15.1 Recall: Conic Duality

Let $K$ be a regular cone, and $K^* = \{ y : y^T x \geq 0, \forall x \in K \}$ be its dual cone

\[
p^* = \min_x c^T x \quad \text{s.t.} \quad Ax \geq_K b \quad \text{(CP)}
\]
\[
d^* = \max_y b^T y \quad \text{s.t.} \quad A^T y = c, y \geq_K 0 \quad \text{(CD)}
\]

- Weak Duality: $p^* \geq d^*
- Strong Duality: If one of (CP), (CD) is bounded and strictly feasible, then $p^* = d^*.
- Optimality Condition: $(x^*, y^*)$ is optimal iff
  (i) primal feasibility: $Ax^* \geq_K b$
  (ii) dual feasibility: $A^T y^* = c, y^* \geq_K 0$
  (iii) zero-duality gap: $c^T x^* - b^T y^* = 0$
15.2 SOCP Duality

Second-order Cone Program (SOCP): when $K$ is the Cartesian product of Lorentz cones, namely, $K = L^{n_1} \times ... \times L^{n_m}$.

The general form of a second-order cone program is

\[
\begin{align*}
\min_x & \quad c^T x \\
\text{s.t.} & \quad \|A_i x - b_i\|_2 \leq d_i^T x - e_i, \quad i = 1, ..., m
\end{align*}
\]

(SOCP–P)

where $c \in \mathbb{R}^n, A_i \in \mathbb{R}^{(n_i-1) \times n}, R_i \in \mathbb{R}^{(n_i-1) \times 1}, e_i \in \mathbb{R}$.

The conic form is

\[
\begin{align*}
\min_x & \quad c^T x \\
\text{s.t.} & \quad \tilde{A}_i x \geq \tilde{b}_i, \quad i = 1, ..., m
\end{align*}
\]

where $\tilde{A}_1 = \begin{bmatrix} A_i^T \\ d_i^T \end{bmatrix}, \tilde{b}_i = \begin{bmatrix} b_i \\ e_i \end{bmatrix}$

**Proposition 15.1** $L^n$ is self-dual, i.e. $(L^n)^* = L^n$

**Proof:**

(i) $L^n \subseteq (L^n)^*$

Suppose $y \in L^n$, we show that $\forall x \in L^n$,

\[y^T x = y_1 x_1 + ... + y_n x_n \geq -\sqrt{\sum_{i=1}^{n-1} y_i^2} \sqrt{\sum_{i=1}^{n-1} x_i^2} + y_n x_n \geq 0\]

where the first inequality is due to Cauchy-Schwarz inequality and the second inequality is because $y_n \geq \sqrt{\sum_{i=1}^{n-1} y_i^2}$ and $x_n \geq \sqrt{\sum_{i=1}^{n-1} x_i^2}$.

(ii) $(L^n)^* \subseteq L^n$

Suppose $y \in (L^n)^*$, we have $y^T x \geq 0, \forall x \in L^n$

If $(y_1, ..., y_{n-1}) = 0$, setting $x = [0, ..., 0, 1] \in L^n$, we get $y^T x = y_n \geq 0$, so $y \in L^n$

If $(y_1, ..., y_{n-1}) \neq 0$, setting $x = [-y_1, ..., -y_{n-1}, \sqrt{\sum_{i=1}^{n-1} y_i^2}] \in L^n$, we get

\[y^T x = \sum_{i=1}^{n-1} y_i^2 + y_n \sqrt{\sum_{i=1}^{n-1} y_i^2} \geq 0 \Rightarrow y_n \geq \sqrt{\sum_{i=1}^{n-1} y_i^2}, \text{ i.e. } y \in L^n\]
Hence, the dual of the SOCP is
\[
\max_{\lambda \in \mathbb{R}^m, u_i \in \mathbb{R}^{n_i-1}, i=1,...,m} \quad \sum_{i=1}^{m} b_i^T u_i + e^T \lambda \\
\text{s.t.} \quad \sum_{i=1}^{m} (A_i^T u_i + d_i \lambda_i) = c \\
\| u_i \|_2 \leq \lambda_i, \quad i = 1, ..., m
\]

\textbf{Remark}: The same dual can be derived from Lagrange duality. The Lagrange function is
\[
L(x, \lambda) = c^T x + \sum_{i=1}^{m} \lambda_i \left( \| A_i x - b_i \|_2 - (d_i^T x - e_i) \right)
\]
The Lagrange dual is given by
\[
\max_{\lambda \geq 0} \min_x L(x, \lambda) \\
= \max_{\lambda \geq 0} \min_x \sum_{i=1}^{m} \lambda_i \left( \| A_i x - b_i \|_2 - (d_i^T x - e_i) \right) \\
= \max_{\lambda \geq 0, \| u_i \|_2 \leq \lambda_i} \min_x \left( c - \sum_{i=1}^{m} (A_i^T u_i + d_i \lambda_i) \right)^T x + \sum_{i=1}^{m} b_i^T u_i + e^T \lambda \\
= \max_{\lambda, u_1, ..., u_m} \sum_{i=1}^{m} b_i^T u_i + e^T \lambda \\
\text{s.t.} \quad \sum_{i=1}^{m} (A_i^T u_i + d_i \lambda_i) = c \\
\| u_i \|_2 \leq \lambda_i, \quad i = 1, ..., m
\]
which is the same as dual derived from conic duality.

15.3 SDP Duality

\textbf{Semidefinite Program (SDP)}: when \( K \) is the positive semidefinite cone.
The general form of a semidefinite program is
\[
\min \quad c^T x \\
\text{s.t.} \quad Ax - B = \sum_{i=1}^{n} x_i A_i - B \succeq 0
\]
The constraint type \( x_1 A_1 + ... + x_n A_n - B \succeq 0 \) is called Linear Matrix Inequality.

\textbf{Proposition 15.2} \( S_+^n \) is self-dual, i.e. \( (S_+^n)^* = S_+^n \)}
Proof:

(i) $S^n_+ \subseteq (S^n_+)^*$
Suppose $Y \succeq 0$. we have $Y = U\Lambda U^T = \sum_{i=1}^n \lambda_i u_i u_i^T$, where $\lambda \geq 0$, $i = 1, ..., m$

$$\forall X \succeq 0, \langle X, Y \rangle = tr(X^TY) = tr(XY) = tr(\sum_{i=1}^n \lambda_i u_i u_i^T Y) = \sum_{i=1}^n \lambda_i tr(u_i^TY u_i) \geq 0$$

Hence, $Y \in (S^n_+)^*$.

(ii) $(S^n_+)^* \subseteq S^n_+$
Suppose $Y \in (S^n_+)^*$, i.e. $tr(XY) \geq 0, \forall x \in S^n_+$. For any $x \in \mathbb{R}^n$, let $X = xx^T$, we have

$$tr(XY) = tr(x x^T Y) = x^TY x \geq 0.$$

Hence, $Y \in S^n_+$.

The dual of SDP is:

$$\max_Y \quad tr(BY)$$

$$\text{s.t. } tr(A_i Y) = c_i \quad i = 1, ..., n$$

$$Y \succeq 0$$

(SDP–D)

Remark: Based on conic duality theorem $(x^*, y^*)$ is optimal primal-dual pair iff

1. $\sum_{i=1}^n x_i A_i \succeq B$ (primal feasibility)
2. $Y \succeq 0, tr(A_i Y) = c_i, i = 1, ..., m$ (dual feasibility)
3. $tr(Y(\sum_{i=1}^n x_i A_i - B)) = 0$, i.e. $Y(\sum_{i=1}^n x_i A_i - B) = 0$

Example: Use SDP duality to show that for any $B \in S^n_+$:

$$\lambda_{max}(B) = \max_{x \in \mathbb{R}^n} \{ x^T B x : \| x \|_2 = 1 \}$$

Note that the right hand side can be rewritten as

$$\max_x \quad tr(Bxx^T)$$

$$\text{s.t. } tr(x x^T) = 1$$

We show this is equivalent to solve the SDP relaxation.

$$\max_X \quad tr(BX)$$

$$\text{s.t. } tr(X) = 1 \quad X \succeq 0$$

(P)
Let \( p = \max_x \{ \text{tr}(Bx^T) : \text{tr}(xx^T) = 1 \} \) and \( \bar{p} = \max_X \{ \text{tr}(BX) : \text{tr}(X) = 1, X \succ 0 \} \). Since the latter is a relaxation, so \( \bar{p} \geq p \). On the other hand, for any \( X \succ 0 \) with \( \text{tr}(X) = 1 \),
\[
X = \sum_{i=1}^n \lambda_i x_ix_i^T, \quad \text{with } \sum_{i=1}^n \lambda_i = 1, \lambda_i \geq 0, i = 1, \ldots, n, \quad \text{and } \| x_i \|_2 = 1, \text{i.e. } \text{tr}(x_ix_i^T) = 1
\]
\[
\text{tr}(BX) = \text{tr}(\sum_{i=1}^n \lambda_i x_ix_i^TB) = \sum_{i=1}^n \lambda_i \text{tr}(Bx_ix_i^T) \leq \max_{i=1,\ldots,n} \text{tr}(Bx_ix_i^T) \leq p
\]
Hence, \( \bar{p} \leq p \). Therefore, \( \bar{p} = p \).

By the SDP duality, the dual of (P) is
\[
\min_x \lambda \\
\text{s.t. } \lambda I - B \succ 0 \quad (D)
\]

Note that \( \lambda I - B \succ 0 \) is equivalent to \( \lambda \geq \lambda_{\max}(B) \).

Since (P) is strictly feasible, \( X = n^{-1}I \) satisfies \( \text{tr}(X) = 1 \) and \( X \succ 0 \), strong duality holds. Therefore,
\[
\max_x \{ x^TBx : \| x \|_2 = 1 \} = \text{Opt}(P) = \text{Opt}(D) = \lambda_{\max}(B)
\]

### 15.4 SDP Relaxations of Non-convex Quadratic Programming

Consider the quadratic constrained quadratic programming:
\[
\text{Opt} = \min \quad x^TQ_0x + 2q_0^Tx + c_0 \\
\text{s.t. } \quad x_i^TQ_ix_i + 2q_i^Tx + c_i \leq 0, \quad 1 \leq i \leq m \quad (QCQP)
\]
Let \( X = \begin{bmatrix} xx^T & x \\ x^T & 1 \end{bmatrix} = \begin{bmatrix} x \end{bmatrix} \begin{bmatrix} x^T \\ 1 \end{bmatrix}^T \) and \( A_i = \begin{bmatrix} Q_i & q_i \\ q_i^T & c_i \end{bmatrix}, i = 0, 1, \ldots, m \)

We can write (QCQP) as
\[
\min_{x,X} \quad \text{tr}(A_0X) \\
\text{s.t. } \quad \text{tr}(A_iX) \leq 0, \quad i = 1, \ldots, m \\
X = \begin{bmatrix} xx^T & x \\ x^T & 1 \end{bmatrix}
\]
**SDP Relaxation.** The following SDP is a relaxation of (QCQP)

\[
\text{Opt}^{\text{SDP}} = \min_X \text{tr}(A_0X) \\
\text{s.t.} \quad \text{tr}(A_iX) \leq 0, \quad i = 1, \ldots, m \\
X \succ 0 \\
X_{n+1,n+1} = 1
\]

(SDP-relaxation)

Therefore, \(\text{Opt}^{\text{SDP}} \leq \text{Opt}\).

**Lagrange Relaxation.** Another relaxation of QCQP is based on Lagrange duality. The Lagrange dual is given by

\[
\max_{\lambda \geq 0} \inf_x \ L(x, \lambda) = \max_{\lambda \geq 0} \inf_x \ x^T Q_0 x + 2q_0^T x + c_0 + x^T \left( \sum_{i=1}^m \lambda_i Q_i \right) x + 2 \left( \sum_{i=1}^m \lambda_i q_i \right)^T x + \sum_{i=1}^m \lambda_i c_i \geq t, \forall x
\]

Note that \(\forall x, x^T A x \geq 0\) is equivalent to \(A \succ 0\). Hence, the Lagrange relaxation is

\[
\text{Opt}^{\text{Lagrange}} = \max_{\lambda \geq 0} \ t \\
\text{s.t.} \quad \left[ Q_0 + \sum_{i=1}^m \lambda_i Q_i \right] \left[ (q_0 + \sum_{i=1}^m \lambda_i q_i)^T \sum_{i=1}^m \lambda_i c_i + c_0 - t \right] \succ 0
\]

(Lagrange relaxation)

By weak duality, we know that \(\text{Opt}^{\text{Lagrange}} \leq \text{Opt}\).

Indeed, one can show that the Lagrange relaxation is the SDP dual to the SDP relaxation.

- If either of them is strictly feasible, then \(\text{Opt}^{\text{Lagrange}} = \text{Opt}^{\text{SDP}}\).
- Otherwise, we always have \(\text{Opt}^{\text{Lagrange}} \leq \text{Opt}^{\text{SDP}} \leq \text{Opt}\).

Therefore, SDP relaxation is always tighter than Lagrange relaxation.
In this lecture, we cover the following topics

- Robust Linear Program
- Example: Robust Portfolio Selection
- Example: Robust Classification

### 16.1 Robust Linear Program

Consider the linear program

$$\min \{ c^T x : Ax \leq b \}$$  \hspace{1cm} (LP)

Assume that the data \((c, A, b)\) of the program are not known exactly and vary in a given uncertainty set \(\mathcal{U}\). The goal is to find a robust solution that is

- feasible for all instances, i.e. \(Ax \leq b, \forall (c, A, b) \in \mathcal{U}\)
- optimal for the worst-case objective, i.e. \(\sup \{ c^T x : (c, A, b) \in \mathcal{U} \}\)

We call the following program the **robust counterpart** of (LP):

$$\min_{x} \left\{ \sup_{(c, A, b) \in \mathcal{U}} c^T x : Ax \leq b, \forall (c, A, b) \in \mathcal{U} \right\}$$  \hspace{1cm} (RC)

or equivalently,

$$\min_{x, t} \{ t : c^T x \leq t \text{ and } Ax \leq b \ \forall (c, A, b) \in \mathcal{U} \}$$

Note that the robust counterpart is a semi-infinite convex optimization program with infinitely many linear inequality constraints. The structure of (RC) depends on the geometry of the uncertainty set \(\mathcal{U}\).

Without loss of generality, let’s consider the robust counterpart in the simple form:

$$\min \ c^T x$$

s.t. \( a_i^T x \leq b_i, \ \forall a_i \in \mathcal{U}_i, i = 1, ..., m \)  \hspace{1cm} (RC)
where $\mathcal{U}_i \subseteq \mathbb{R}^n$ is the uncertainty set of $a_i$.

**Polyhedral Uncertainty**

Consider the situation where the uncertainty set is a polyhedron:

$$\mathcal{U}_i = \{a_i : D_i a_i \leq d_i\} \quad i = 1, ..., m$$

The (RC) becomes

$$\min c^T x \quad \text{s.t.} \quad \max_{a_i \in \mathcal{U}_i} a_i^T x \leq b_i$$

By LP duality, we know that

$$\max_{a_i} a_i^T x \iff \min_{p_i} d_i^T p_i \quad \text{s.t.} \quad D_i^T p_i = x, p_i \geq 0$$

Hence, (RC) is equivalent to

$$\min_{x, p_1, ..., p_m} c^T x \quad \text{s.t.} \quad d_i^T p_i \leq b_i, i = 1, ..., m$$

$$D_i^T p_i = x, i = 1, ..., m$$

$$p_i \geq 0, i = 1, ..., m$$

which is a linear program.

**Ellipsoidal Uncertainty**

Consider the situation when the uncertainty set is an ellipsoid

$$\mathcal{U}_i = \{\bar{a}_i + P_i u : \|u\|_2 \leq 1\}, \quad i = 1, ..., m$$

where $\bar{a}_i \in \mathbb{R}^n, P_i \in \mathbb{R}^{n \times n}$.

The (RC) becomes

$$\min_x c^T x \quad \text{s.t.} \quad \max_{a_i \in \mathcal{U}_i} a_i^T x \leq b_i$$

Note that

$$\max_{a_i \in \mathcal{U}_i} a_i^T x = \max_{\|u\|_2 \leq 1} \bar{a}_i^T x + u^T P_i^T x = \bar{a}_i^T x + \max_{\|u\|_2 \leq 1} u^T P_i^T x = \bar{a}_i^T x + \|P_i^T x\|_2$$
Hence, (RC) is equivalent to
\[
\min_x \quad c^T x \\
\text{s.t.} \quad \|P_ix\|_2 \leq b_i - \bar{a}_i^T x
\]
which is an SOCP.

16.2 Examples

16.2.1 Example I: Robust Portfolio Selection

Consider a classical portfolio optimization problem with \(n\) assets. Let \(x = (x_1, \ldots, x_n)\) be the portfolio vector.

1. Assume that returns \(r_i, i = 1, \ldots, n\) are exactly known. To maximize the return leads to the optimization problem.

   \[
   \max_x \quad r^T x \quad \iff \quad \max_{x, \gamma} \quad \gamma \\
   \text{s.t.} \quad \sum_{i=1}^n x_i = 1 \quad \text{s.t.} \quad r^T x \geq \gamma \\
   \quad x \geq 0 \quad \sum_{i=1}^n x_i = 1, x \geq 0
   \]

   which leads to a highly non-robust solution.

2. Assume the returns are know within an ellipsoid:

   \[\mathcal{U} = \left\{ \hat{r} + \rho \hat{\Sigma}^{1/2} u : \quad \|u\|_2 \leq 1 \right\}\]

   where \(\hat{r}\) and \(\hat{\Sigma}\) are the empirical mean and covariance matrix.

   The robust portfolio problem is

   \[
   \max_x \min_{r \in \mathcal{U}} \gamma^T x, \text{ s.t. } x \geq 0, \sum_{i=1}^n x_i = 1
   \]

   which is equivalent as an SOCP:

   \[
   \max_x \quad r^T x - \rho \|\hat{\Sigma}^{1/2} x\|_2, \text{ s.t. } x \geq 0, 1^T x = 1
   \]

   This can be interpreted as a return-risk tradeoff.
3. Assume that returns are random variables with mean $\hat{r}$ and covariance $\hat{\Sigma}$. Consider the chance constrained program:

$$\max_{x, \gamma} \quad \gamma$$
$$\text{s.t.} \quad P(r^T x \geq \gamma) \geq 1 - \epsilon$$
$$\sum_{i=1}^{n} x_i = 1$$
$$x \geq 0$$

Note that

$$P(r^T x \geq \gamma) \geq 1 - \epsilon \equiv P(Z \geq \frac{\gamma - \hat{r}^T x}{\sqrt{x^T \hat{\Sigma} x}}) \geq 1 - \epsilon,$$ where $Z \sim N(0, 1)$

$$\Leftrightarrow P(Z \leq \frac{\hat{r}^T x - \gamma}{\sqrt{x^T \hat{\Sigma} x}}) \geq 1 - \epsilon$$

$$\Leftrightarrow \frac{\hat{r}^T x - \gamma}{\sqrt{x^T \hat{\Sigma} x}} \geq \Phi^{-1}(1 - \epsilon)$$

$$\Leftrightarrow \hat{r}^T x - \Phi^{-1}(1 - \epsilon)\|\hat{\Sigma}^{1/2} x\|_2 \geq \Gamma$$

Hence, the chance constrained program is equivalent to

$$\max_x \quad \hat{r}^T x - \Phi^{-1}(1 - \epsilon)\|\hat{\Sigma}^{1/2} x\|_2,$$ s.t. $x \geq 0, 1^T x = 1$

**Remark** Robust LP with ellipsoidal uncertainty set are closed related to chance constraints of a stochastic model.

### 16.2.2 Example 2: Robust Classification

Consider the support vector machine model for binary classification:

$$\min_{\omega, b, \epsilon} \quad \sum_{i=1}^{m} \epsilon_i$$
$$\text{s.t.} \quad y_i(\omega^T x_i + b) \geq 1 - \epsilon_i$$
$$\epsilon \geq 0$$

(SVM)

where $(x_i, y_i), i = 1, ..., m$ are data points with $x_i \in \mathbb{R}^n, y_i \in \{\pm 1\}$.

1. Assume that the feature vector $x_i$ are subject to spherical uncertainty

$$x_i \in X_i = \{\hat{x}_i + \rho u : \|u\|_2 \leq 1\}$$
The robust counterpart of (SVM) simplifies to a SOCP

\[
\min_{\omega, b, \epsilon} \sum_{i=1}^{m} \epsilon_i \\
\text{s.t.} \quad y_i (\omega^T \hat{x}_i + b) - \rho \|\omega\|_2 \geq 1 - \epsilon_i \\
\epsilon_i \geq 0
\]

which is very similar to the classical SVM with $\ell_2$-norm regularization:

\[
\min_{\omega, b} \sum_{i=1}^{m} \max \left( 1 - y_i (\omega^T x_i + b), 0 \right) + \lambda \|\omega\|_2^2
\]

2. Assume that the feature vector $x_i$ are subject to box uncertainty

\[
x_i \in X_i = \{ \hat{x}_i + \rho u : \|u\|_{\infty} \leq 1 \}
\]

The robust counterpart of (SVM) becomes

\[
\min_{\omega, b} \sum_{i=1}^{m} \max \left( 1 - y_i (\omega^T x_i + b), 0 \right) + \rho \|\omega\|_{1,0}
\]

which is very similar to the classical SVM with $\ell_1$-norm regularization.

**Remark** Robust optimization has a close connection to the regularization technique in machine learning.
In this lecture, we cover the following topics

- Path-following Scheme
- Self-concordant Functions

Reference:

Nemirovski, *Interior Point Polynomial Time Methods in Convex Programming*, 2004, Chapter 1
Nesterov, *Introductory Lectures on Convex Optimization*, 2004, Chapter 4.1

17.1 Historical Notes

- 1947: Dantzig, Simplex method for LP
- 1973: Klee and Minty proved that Simplex Method is not a polynomial-time algorithm
- mid-1970s: Shor, Nemirovski and Yudin, Ellipsoid method for linear and convex programs
- 1979: Khachiyan proved the polynomial-time solvability of LP
- 1984: Karmarkar, polynomial algorithm (potential reduction interior point method) for LP
  (1967): Dikin, affine scaling algorithm (simplification of Karmarkar’s algorithm) for LP
- late-1980s: Renegar, Gonzaga, path-following interior point method for LP
- 1988: Nesterov and Nemirovski, extend interior point method for convex programs

17.2 Path following Scheme

We intend to solve a general convex program

\[
\min_x \ f(x) \\
\text{s.t.} \ g_i(x) \leq 0, \ i = 1, ..., m
\]

where \( f, g_i \) are twice continuously differentiable convex functions.
Denote $X = \{ x : g_i(x) \leq 0, \forall i = 1, \ldots, m \}$ as the feasible domain. Assume Slater condition holds and $X$ is bounded, so $X$ is a compact convex set with non-empty interior.

**Barrier Method**: solve a series of unconstrained problems

$$\min_x \ tf(x) + F(x) \quad (P_t)$$

where $t > 0$ is a penalty parameter and $F(x)$ is a **barrier function** that satisfies:

- $F : \text{int}(X) \to \mathbb{R}$ and $F(x) \to +\infty$ as $x \to \partial(X)$
- $F$ is smooth (twice continuously differentiable) and convex
- $F$ is non-degenerate, i.e. $\nabla^2 F(x) \succ 0, \forall x \in \text{int}(X)$

Note that for any $t > 0$, $(P_t)$ has a unique solution in the interior of $X$.

Denote

$$x^*(t) = \arg \min_x \ tf(x) + F(x)$$

The path $\{x^*(t), t > 0\}$ is called the **central path**. We have

$$x^*(t) \to x^*, \text{ as } t \to \infty$$

To implement the above path-following scheme, need to specify:

1. the barrier function $F(x)$:
   - use self-concordant barriers, e.g. $F(x) = -\sum_{i=1}^{m} \log(-g_i(x))$
2. the method to solve unconstrained minimization problems $(P_t)$:
   - use Newton method
3. the policy to update the penalty parameter $t$.

### 17.3 Self-concordant Function

**Definition 17.1** A function $f : \mathbb{R} \to \mathbb{R}$ is **self-concordant** if $f$ is convex and

$$|f'''(x)| \leq \kappa f''(x)^{3/2}, \forall x \in \text{dom}(f)$$

for some constant $\kappa \geq 0$.

When $\kappa = 2$, $f$ is called a **standard** self-concordant function.

**Examples:**
- Logarithmic function: \( f(x) = -\ln(x), x > 0 \) is standard self-concordant:
  \[
  f'(x) = -\frac{1}{x}, \quad f''(x) = \frac{1}{x^2}, \quad f'''(x) = -\frac{2}{x^3}, \quad \frac{|f'''(x)|}{f''(x)^{3/2}} = 2
  \]
- Linear function: \( f(x) = cx \) is self-concordant with constant \( \kappa = 0 \):
  \[
  f'(x) = c, \quad f''(x) = 0, \quad f'''(x) = 0
  \]
- Convex quadratic function: \( f(x) = \frac{a}{2}x^2 + bx + c \) (\( a > 0 \)) is self-concordant with \( \kappa = 0 \):
  \[
  f'(x) = ax + b, \quad f''(x) = a, \quad f'''(x) = 0
  \]
- Exponential function: \( f(x) = e^x \) is not self-concordant:
  \[
  f'(x) = f''(x) = f'''(x) = e^x, \quad \frac{|f'''(x)|}{f''(x)^{3/2}} = e^{x/2} \to +\infty \text{ as } x \to -\infty
  \]
- Power functions:
  \[
  f(x) = 1 \cdot x^p (p > 0), \quad f(x) = |x|^p (p > 2), \quad f(x) = x^{2p} (p > 2)
  \]
  are not self-concordant.

**Remark:** Self-concordant function is **affine-invariant:** If \( f(x) \) is self-concordant, \( \tilde{f}(y) = f(ay+b) \) is also self-concordant with the same constant.

**Proof:** It is easy to see that \( \tilde{f} \) is convex and
\[
\frac{\tilde{f}'''(y)}{\tilde{f}''(y)^{3/2}} = \frac{|a^3f'''(ay+b)|}{[a^2f''(ay+b)]^{3/2}} = \frac{f'''(ay+b)}{f''(ay+b)^{3/2}} \leq \kappa
\]

When extending to a function \( f \) defined on \( \mathbb{R}^n \) and \( f \in C^3(\mathbb{R}^n) \), we say \( f \) is self-concordant if it is self-concordant along every line, namely, \( \forall x \in \text{dom}(f), h \in \mathbb{R}^n, \phi(t) = f(x + th) \) is self-concordant with some constant \( \kappa \geq 0 \).

Denote
\[
D^k f(x)[h_1, \ldots, h_k] = \frac{\partial^k}{\partial t_1 \ldots \partial t_k} |_{t_1 = \ldots = t_k = 0} f(x + t_1 h_1 + \ldots + t_k h_k)
\]
as the k-th differential of \( f \) taken at \( x \) along the directions \( h_1, \ldots, h_k \) e.g.
\[
Df(x)[h] = f'(0) = \langle \nabla f(x), h \rangle
\]
\[
D^2 f(x)[h, h] = f''(0) = \langle \nabla^2 f(x) h, h \rangle
\]
Definition 17.2 We say a function \( f : \mathbb{R}^n \to \mathbb{R} \) is self-concordant if
\[
D^3 f(x)[h,h,h] \leq \kappa (D^2 f(x)[h,h])^{3/2}, \forall x \in \text{dom}(f), h \in \mathbb{R}^n
\]
for some constant \( \kappa \geq 0 \)

Operations Preserving Self-Concordance

Proposition 17.3

1. (Affine invariant) If \( f(y) \) is self-concordant with constant \( \kappa \), then the function \( \tilde{f}(x) = f(Ax + b) \) is also self-concordant with constant \( \kappa \).

2. (Summation) If \( f_1(x) \) and \( f_2(x) \) are self-concordant with constants \( \kappa_1, \kappa_2 \), then the function \( \tilde{f}(x) = f_1(x) + f_2(x) \) is self-concordant with constant \( \kappa = \max \{ \kappa_1, \kappa_2 \} \)

3. (Scaling) If \( f(x) \) is self-concordant with constant \( \kappa \), and \( \alpha > 0 \) then the function \( \alpha f(x) \) is also self-concordant with \( \kappa = \frac{\kappa}{\sqrt{\alpha}} \)

Proof: Exercise!

Hence, it is straightforward to see that

Corollary 17.4 \( f(x) = -\sum_{i=1}^{m} \ln(b_i - a_i^T x) \) is standard self-concordant on \( \text{int}(X) \), where \( X = \{ x : a_i^T x \leq b_i, i = 1, ..., m \} \)

Remark: Indeed, under regular conditions, self-concordant functions are barrier functions:

- If \( \text{dom}(f) \) contains no straight line, then \( \nabla^2 f(x) \) is non-degenerate (strictly convex)
- If \( f \) is closed convex, then \( f(x_k) \to +\infty \) if \( \{ x_k \} \subseteq \text{dom}(f) \) and \( x_k \to \bar{x} \in \partial(\text{dom}(f)) \)
In this lecture, we cover the following topics:

- Classical Newton Method for Unconstrained Minimization
- (Damped) Newton Method for Self-concordant Functions

Reference:
Nesterov, Introductory Lectures on Convex Optimization, 2004, Chapter 1.2.4

18.1 Classical Newton Method

Consider the unconstrained minimization

$$\min_{x \in \mathbb{R}^n} f(x)$$

where $f(x)$ is twice continuously differentiable (not necessarily convex) on $\mathbb{R}^n$.

Newton Method:

$$x_{k+1} = x_k - [\nabla^2 f(x_k)]^{-1} \nabla f(x_k), k = 0, 1, 2, ...$$

The direction $d_k = -\nabla^2 f(x_k)^{-1} \nabla f(x_k)$ is called Newton’s direction.

Interpretation: The Newton method can be treated as

- Minimizing quadratic approximation of $f$: From Taylor’s expression, we have:

$$f(x + h) = f(x) + \nabla f(x)^T h + \frac{1}{2} h^T \nabla^2 f(x) h + o(\|h\|^2)$$

The iteration can be viewed as minimizing the quadratic approximation of $f$:

$$x_{k+1} = \min_{x} \left\{ f(x_k) + \nabla f(x_k)^T (x - x_k) + \frac{1}{2} (x - x_k)^T \nabla^2 f(x_k) (x - x_k) \right\}$$
• **Solving linearized optimality condition**: From first-order optimality condition: $\nabla f(x) = 0$.

Note that

$$\nabla f(x + h) \approx \nabla f(x) + \nabla^2 f(x) h$$

The iteration can also be viewed as solving linearized optimality condition

$$\nabla f(x_k) + \nabla^2 f(x_k)(x - x_k) = 0$$

**Remark (Newton method vs Gradient Descent)**

$$x_{k+1} = x_k - \nabla^2 f(x_k)^{-1} \nabla f(x_k) \quad \text{(Newton)}$$

$$x_{k+1} = x_k - \gamma_k \nabla f(x_k) \quad \text{(GD)}$$

Firstly, unlike Gradient Descent, Newton’s direction is not necessarily a descent direction: for $\nabla f(x) \neq 0$,

$$\nabla f(x)^T d = -\nabla f(x)^T \nabla^2 f(x) \nabla f(x) \nabla f(x) \neq 0$$

Secondly, Newton method is a second-order method and requires relatively high computational cost comparing to GD.

**Remark (Convergence)**

(i) Newton method can break down if $\nabla^2 f(x)$ is degenerate.

(ii) When $f$ is quadratic and non-degenerate, Newton method converges in one step.

(iii) The method may diverge even for a nice strongly convex function.

(iv) If started close enough to a strict local minimum, the method can converge very fast.

**Example:** consider the convex function $f(x) = \sqrt{1 + x^2}$

One can easily compute that $x^* = 0$, and

$$f'(x) = \frac{x}{\sqrt{1 + x^2}}, \quad f''(x) = \frac{1}{(1 + x^2)^{3/2}}$$

The Newton method works as

$$x_{k+1} = x_k - (1 + x_k^2)^{3/2} \frac{x_k}{\sqrt{1 + x_k^2}} = -x_k^3$$

Note that

- if $|x_0| < 1$, the method converges extremely fast
- if $|x_0| = 1$, the method oscillates between 1 and -1
- if $|x_0| > 1$, the method diverges.
18.2 Classical Analysis

Theorem 18.1 (Local quadratic convergence of Newton method) Assume that

- \( f \) has a Lipschitz Hessian: \( \| \nabla^2 f(x) - \nabla^2 f(y) \|_2 \leq M \| x - y \|_2 \) for \( M > 0 \).
- \( f \) has a strict local minimum \( x^* \): \( \nabla^2 f(x^*) \succcurlyeq \mu I \), with \( \mu > 0 \).
- The initial point \( x_0 \) is close enough to \( x^* \): \( \| x_0 - x^* \|_2 \leq \frac{\mu}{2M} \).

Then Newton method is well-defined and converges to \( x^* \) at a quadratic rate:

\[
\| x_{k+1} - x^* \|_2 \leq \frac{M}{\mu} \| x_k - x^* \|_2^2
\]

Note that for a symmetric matrix \( A \), \( \| A \|_2 := \sup_{x \neq 0} \left\{ \frac{\| Ax \|_2}{\| x \|_2} \right\} = \max_k |\lambda_k(A)| \). We first prove the following simple useful lemma.

Lemma 18.2 Suppose \( f \) has Lipschitz Hessian with constant \( M \), then for any \( x, y \),

\[
\| \nabla f(y) - \nabla f(x) - \nabla^2 f(x)(y - x) \|_2 \leq \frac{M}{2} \| y - x \|_2^2.
\]

Proof: This is because by basic calculus

\[
\nabla f(y) - \nabla f(x) = \int_0^1 \nabla^2 f(x + t(y - x))(y - x)dt
\]

\[
\| \nabla f(y) - \nabla f(x) - \nabla^2 f(x)(y - x) \|_2 = \| \int_0^1 \nabla^2 f(x + t(y - x))(y - x)dt - \nabla^2 f(x)(y - x) \|_2
\]

\[
= \| \int_0^1 \left[ \nabla^2 f(x + t(y - x)) - \nabla^2 f(x) \right] (y - x)dt \|_2
\]

\[
\leq \int_0^1 Mt \| y - x \|_2^2 dt
\]

\[
= \frac{M}{2} \| y - x \|_2^2
\]

Now we are ready to prove the main theorem.

Proof: We have

\[
x_{k+1} - x^* = x_k - x^* - [\nabla^2 f(x_k)]^{-1} \nabla f(x_k)
\]

\[
= [\nabla^2 f(x_k)]^{-1} [\nabla^2 f(x_k)(x_k - x^*) - \nabla f(x_k)]
\]

\[
= [\nabla^2 f(x_k)]^{-1} [\nabla f(x^*) - \nabla f(x_k) - \nabla^2 f(x_k)(x^* - x_k)]
\]
The last equality is due to the fact that $\nabla f(x^*) = 0$ by optimality condition. Hence,

$$
\| x_{k+1} - x^* \|_2 \leq \| [\nabla^2 f(x_k)]^{-1} \|_2 \| \nabla f(x^*) - \nabla f(x_k) - \nabla^2 f(x_k)(x^* - x_k) \|_2 \\
\leq \| [\nabla^2 f(x_k)]^{-1} \|_2 \frac{M}{2} \| x_k - x^* \|_2^2
$$

The last inequality is due to the previous lemma.

We show now by induction that $\| x_k - x^* \|_2 \leq \frac{\mu}{M^2}$.

Assume $\| x_k - x^* \|_2 \leq \frac{\mu}{M^2}$, then

$$
\| \nabla^2 f(x_k) - \nabla^2 f(x^*) \|_2 \leq M \| x_k - x^* \|_2 \leq \frac{\mu}{2}
$$

which implies that $-\frac{\mu}{2} I \preceq \nabla^2 f(x_k) - \nabla^2 f(x^*) \preceq \frac{\mu}{2} I$.

Hence,

$$
\nabla^2 f(x_k) \succeq \frac{\mu}{2} I
$$

which implies that $\| [\nabla^2 f(x_k)]^{-1} \|_2 \leq \frac{2}{\mu}$. This leads to

$$
\| x_{k+1} - x^* \|_2 \leq \frac{M}{\mu} \| x_k - x^* \|_2^2
$$

and $\| x_{k+1} - x^* \|_2 \leq \frac{\mu}{2M}$, which concludes the proof.

Note that the local convergence holds for any unconstrained minimization regardless of convex or not. When $f$ is strongly convex, the analysis is even simpler.

**Theorem 18.3** Assume that

- $f$ has a Lipschitz Hessian $\| \nabla^2 f(x) - \nabla f(y) \|_2 \leq L \| x - y \|_2$
- $f$ is $\mu$-strongly convex, i.e., $\nabla^2 f(x) \succeq \mu I, \forall x$
- The initial point $x_0$ satisfies $\| \nabla f(x_0) \|_2 < \frac{2\mu^2}{M}$

Then the gradient converges to zero quadratically

$$
\| \nabla f(x_{k+1}) \|_2 \leq \frac{M}{2\mu^2} \| \nabla f(x_k) \|_2^2
$$

**Proof**: Setting $y = x_{k+1}, x = x_k$ in the previous lemma, we have

$$
\| \nabla f(x_{k+1}) \|_2 \leq \frac{M}{2} \| [\nabla^2 f(x_k)]^{-1} \nabla f(x_k) \|_2 \\
\leq \frac{M}{2} \| [\nabla^2 f(x_k)]^{-1} \|_2^2 \cdot \| \nabla f(x_k) \|_2 \\
\leq \frac{M}{2\mu^2} \| \nabla f(x_k) \|_2^2
$$
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Remark (Complexity)  The quadratic convergence implies that:

\[
\frac{M}{\mu} \| x_k - x^* \|_2 \leq \left( \frac{M}{\mu} \| x_{k-1} - x^* \|_2 \right)^2 \\
\leq \left( \frac{M}{\mu} \| x_{k-2} - x^* \|_2 \right)^4 \\
\leq \ldots \\
\leq \left( \frac{M}{\mu} \| x_0 - x^* \|_2 \right)^{2k} \\
\leq \left( \frac{1}{2} \right)^{2k}
\]

Hence, \( \| x_k - x^* \|_2 < \frac{\mu}{M} 2^{-2k} \)

To achieve an accuracy \( \epsilon \), i.e. \( \| x_k - x^* \|_2 \leq \epsilon \), the number of iterations

\[
k \geq \log_2 \log_2 \left( \frac{M}{\mu \epsilon} \right)
\]

Remark (Affine Invariance)  The Newton method is invariant w.r.t. affine transformation of variables.

Let \( A \) be a non-singular matrix consider the function

\[
\hat{f}(y) = f(Ay)
\]

The Newton step for \( f \) and \( \hat{f} \) are

\[
x_{k+1} = x_k - \left[ \nabla^2 f(x_k) \right]^{-1} \nabla f(x_k) \\
y_{k+1} = y_k - \left[ \nabla^2 \hat{f}(y_k) \right]^{-1} \nabla \hat{f}(y_k)
\]

Let \( y_0 = A^{-1}x_0 \), then \( y_k = A^{-1}x_k \). This can be shown by induction

\[
y_{k+1} = y_k - \left[ A^T \nabla^2 f(Ay_k) A \right]^{-1} \left[ A^T \nabla f(Ay_k) \right] \\
= A^{-1}x_k - \left[ A^T \nabla^2 f(x_k) A \right]^{-1} A^T \nabla f(x_k) \\
= A^{-1}x_k - A^{-1} \nabla^2 f(x_k)^{-1} \nabla f(x_k) \\
= A^{-1}x_{k+1}
\]

In other words, Newton’s method follow the same trajectory in the ‘x-space’ and ‘y-space’. Hence, the region of quadratic convergence should not depend on the Euclidean metric.

However, in the classical analysis, the assumption and the measure of error, e.g. the Lipschitz continuity of Hessian, depend heavily on the Euclidean metric and is not affine invariant. A natural remedy is to assume self-concordance. Self concordant function are especially well suited for Newton method.
In this lecture, we cover the following topics

- Properties of Self-Concordant Functions
- Minimization of Self-Concordant Functions
- (Damped) Newton Method of Self-Concordant Functions

Reference:
Nesterov, Introductory Lectures on Convex Optimization, 2004, Chapter 4.1.4 - 4.1.5

19.1 Properties of Self-Concordant Functions

Let $f(x)$ be a standard self-concordant function, i.e. $\forall x \in \text{dom}(f), h \in \mathbb{R}^n$

$$\left| D^3 f(x)[h, h, h] \right| \leq 2 \left( D^2 f(x)[h, h] \right)^{3/2}$$

i.e.

$$\left| \frac{d^3}{dt^3} \Big|_{t=0} f(x + th) \right| \leq 2 \left( \frac{d^2}{dt^2} \Big|_{t=0} f(x + th) \right)^{3/2}$$

Definition 19.1 We define the local norm of $h$ at $x \in \text{dom}(f)$ as

$$\| h \|_x = \sqrt{h^T \nabla^2 f(x) h}$$

We state below a basic inequality without proof, for standard self-concordant function $f$, it holds that

$$\left| D^3 f(x)[h_1, h_2, h_3] \right| < 2 \| h_1 \|_x \cdot \| h_2 \|_x \cdot \| h_3 \|_x$$

Remark ("Lipschitz continuity") at a high level,

$$\left| \frac{d}{dt} \Big|_{t=0} D^2 f(x + t\delta)[h, h] \right| \leq 2 \| \delta \|_x \cdot \| D^2 f(x)[h, h] \|_x$$

The second derivative is relatively Lipschitz continuous w.r.t. the local norm defined by $f$. 

19-1
For instance, when \( f \) is self-concordant on \( \mathbb{R} \) and strictly convex,

\[
\frac{|f'''(x)|}{|f''(x)|^{3/2}} \leq 1 \implies \left| \frac{d}{dx} [f''(x)^{-1/2}] \right| \leq 1
\]
\[
\implies -y \leq \int_{0}^{y} \frac{d}{dx} [f''(x)]^{-1/2} dx \leq y
\]
\[
\implies -y \leq \frac{1}{\sqrt{f''(y)}} - \frac{1}{\sqrt{f''(0)}} \leq y
\]

Simplifying the above terms, we arrive at

\[
\frac{f''(0)}{(1 + y \sqrt{f''(0)})^2} \leq f''(y) \leq \frac{f''(0)}{(1 - y \sqrt{f''(0)})^2}, \quad \forall 0 \leq y < \sqrt{f''(0)}
\]

Note that the Hessian is nearly proportional to \( f''(0) \) around a neighborhood of \( x = 0 \).

Moreover, if we integrate (\( \ast \)) and integrate again on both sides (e.g. lower bound):

\[
\frac{|f'''(x)|}{|f''(x)|^{3/2}} \leq 1 \implies \sqrt{f''(0)} - \frac{\sqrt{f''(0)}}{1 + y \sqrt{f''(0)}} \leq f'(y) - f'(0)
\]
\[
\implies y \sqrt{f''(0)} - \ln(1 + y \sqrt{f''(0)}) \leq f(y) - f(0) - f'(0)y
\]

Rewriting the above equations and considering both sides, we arrive at: \( \forall 0 \leq y < \sqrt{f''(0)} \)

\[
\frac{(y \sqrt{f''(0)})^2}{1 + y \sqrt{f''(0)}} \leq y(f'(y) - f'(0)) \leq \frac{(y \sqrt{f''(0)})^2}{1 - y \sqrt{f''(0)}}
\]

and

\[
y \sqrt{f''(0)} - \ln(1 + y \sqrt{f''(0)}) \leq f(y) - f(0) - f'(0)y \leq -y \sqrt{f''(0)} - \ln(1 - y \sqrt{f''(0)})
\]

More generally, for self-concordant functions in \( \mathbb{R}^n \), similar results hold (see, Nesterov, 2004 Theorem 4.1.6-4.1.8) . We provide the results below without providing the proofs.

**Definition 19.2 (Dikin Ellipsoid)**

\[
W_r(x) = \{ y : \| y - x \|_x \leq 1 \}
\]
\[
W_r^0(x) = \{ y : \| y - x \|_x < 1 \}
\]

**Theorem 19.3** For \( x \in \text{dom}(f) \), we have \( W_r^0(x) \subseteq \text{dom}(f) \)

The Dikin ellipsoid of any point in the domain is contained in the domain. More critically, the self-concordant function behaves nicely within this Dikin ellipsoid.
Theorem 19.4 (Hessian of self-concordant function) For \( x \in \text{dom}(f) \), we have
\[
(1 - \| y - x \|_x^2)^2 \nabla^2 f(x) \preceq \nabla^2 f(y) \preceq (1 - \| y - x \|_x)^{-2} \nabla^2 f(x), \quad \forall y \in W_1^0(x)
\]

Theorem 19.5 (Gradient of self-concordant function) For \( x \in \text{dom}(f) \), we have
\[
\| y - x \|_x^2 \leq \langle \nabla f(y) - \nabla f(x), y - x \rangle \leq \| y - x \|_x^2, \quad \forall y \in W_1^0(x)
\]

Theorem 19.6 (Linear approximation of self-concordant function)

1. \( \forall x, y \in \text{dom}(f) \), we have
\[
f(y) \geq f(x) + \langle f'(x), y - x \rangle + \omega(\| y - x \|_x)
\]

2. \( \forall x \in \text{dom}(f), y \in W_2^0(1) \), we have
\[
f(y) \leq f(x) + \langle f'(x), y - x \rangle + \omega_*(\| y - x \|_x)
\]

where \( \omega(t) = t - \ln(1 + t) \) and \( \omega_*(t) = -t - \ln(1 - t) \) is the conjugate.

Proof: See Theorem 4.1.6, 4.1.7, 4.1.8 in (Nesterov, 2004).

19.2 Minimizing Self-Concordant Functions

Consider the unconstrained minimization
\[
\min_{x \in \mathbb{R}^n} f(x)
\]
where \( f \) is a standard self-concordant and non-degenerate (i.e. \( \nabla^2 f(x) \succeq 0 \)) function. Note that the problem is not necessarily solvable, e.g. \( f(x) = -\ln(x) \).

Definition 19.7 (Newton’s decrement) The quantity
\[
\lambda_f(x) = \sqrt{\nabla f(x) \nabla^2 f(x)^{-1} \nabla f(x)}
\]
is called Newton decrement.

Remark: The Newton decrement can be interpreted as

1. The decrease of the second order Taylor expansion after a Newton step:
\[
f(x) - \min_h \left\{ f(x) + h^T \nabla f(x) + \frac{1}{2} h^T \nabla^2 f(x) h \right\} = \frac{1}{2} \lambda_f^2(x)
\]
2. The conjugate local norm of $\nabla f(x)$:

$$
\| \nabla f(x) \|_{x,*} = \max \{ \nabla f(x)^T y, \| y \|_x \leq 1 \} = \| [\nabla^2 f(x)]^{-1/2} \nabla f(x) \|_2 = \lambda_f(x)
$$

3. The local norm of Newton’s direction $d(x) = -\nabla^2 f(x)^{-1} \nabla f(x)$:

$$
\| d(x) \|_x = \lambda_f(x)
$$

**Theorem 19.8** Assume $\lambda_f(x_0) < 1$ for some $x_0 \in \text{dom}(f)$. Then there exists a unique minimizer of $f$.

**Proof:** It suffices to show that the level set $\{ y : f(y) \leq f(x_0) \}$ is bounded. Since

$$
f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \omega(\| y - x \|_x)
$$

$$
\geq f(x) - \| \nabla f(x) \|_{x,*} \cdot \| y - x \|_x + \omega(\| y - x \|_x)
$$

$$
= f(x) - \lambda_f(x) \cdot \| y - x \|_x + \omega(\| y - x \|_x)
$$

we have

$$
f(y) \leq f(x_0) \implies \frac{\omega(\| y - x_0 \|_{x_0})}{\| y - x_0 \|_{x_0}} \leq \lambda_f(x_0) < 1
$$

Notice the function $\phi(t) = \frac{\omega(t)}{t} = 1 - \frac{1}{t} \ln(1+t)$ is strictly increasing in $t \geq 0$. Hence, $\| y - x_0 \|_{x_0} \leq t^*$ for some $t^*$. This implies that the level set must be bounded.

**Remark:** Note that for self-concordant functions, local condition such as $\lambda_f(x_0) < 1$ provides some global information on $f$.

**Example:** Consider the self-concordant function $f(x) = \epsilon x - \ln(x)$ with $\text{dom}(f) := \{ x : x > 0 \}$.

$$
\lambda_f(x) = \sqrt{(\epsilon - \frac{1}{x})(\frac{1}{x^2})^{-1}(\epsilon - \frac{1}{x})} = |1 - \epsilon x|
$$

When $\epsilon \leq 0$, $\lambda_f(x) \geq 1$, and the function is unbounded below and there does not exist a minimizer.

When $\epsilon > 0$, $\lambda_f(x) < 1$, for $x \in (0, \frac{2}{\epsilon})$, there exists a unique minimizer $x^* = \frac{1}{\epsilon}$.
In this lecture, we cover the following topics

- Newton method for self-concordant functions
- Damped Newton method

Reference: Nesterov, Introductory Lectures on Convex Optimization, 2004, Chapter 4.1.5

20.1 Recall

In the last lecture, we discussed the unconstrained minimization of self-concordant function

\[
\min_x f(x)
\]

where \( f(x) \) is standard self-concordant and non-degenerate.

Dikin ellipsoid: \( W^0_r(x) = \{ y : \| y - x \|_x < r \} \)

For standard self-concordant function, \( W^0_r(x) \subseteq \text{dom}(f), \forall x \in \text{dom}(f) \). Moreover, \( f \) has some nice behavior inside the Dikin ellipsoid. \( \forall y : \| y - x \|_x = \gamma < 1 \), it holds

1. \( (1 - r)^2 \nabla^2 f(x) \preceq \nabla^2 f(y) \preceq \frac{1}{(1 - \gamma)^2} \nabla^2 f(x) \)
2. \( \frac{\gamma^2}{1 + \gamma} \leq \langle \nabla f(y) - \nabla f(x), y - x \rangle \leq \frac{\gamma^2}{1 - \gamma} \)
3. \( \omega(\gamma) \leq f(y) - f(x) - \langle \nabla f(x), y - x \rangle \leq \omega_*(\gamma) \)

where \( \omega(t) = t - \ln(1 + t), \omega_*(t) = -t - \ln(1 - t) \).

Newton’s decrement: \( \lambda_f(x) = \sqrt{\nabla f(x)^T [\nabla^2 f(x)]^{-1} \nabla f(x)} \)

- Note that \( \lambda_f(x) = \| \nabla f(x) \|_{x,*} = \| d(x) \|_x \), where \( d(x) = [\nabla^2 f(x)]^{-1} \nabla f(x) \)
- If \( \lambda_f(x) < 1 \), the point \( x_+ = x - d(x) \in \text{dom}(f) \)
- If \( x^* \) is a minimizer of \( f \), then \( \lambda_f(x^*) = 0 \)
- \( \lambda_f(x_0) < 1 \) for some \( x_0 \in \text{dom}(f) \), then \( f \) has a unique minimizer.
20.2 Newton Method for Self-concordant Function

Basic Newton method: initialize $x_0 \in \text{dom}(f)$ and update via

$$x_{k+1} = x_k - [\nabla^2 f(x_k)]^{-1} \nabla f(x_k), \ k = 0, 1, 2, \ldots$$

Unlike the classical analysis, we should adopt different measures of error that are independent of Euclidean metric. For instance,

- Function gap: $f(x_k) - f(x^*)$
- Newton’s decrement: $\lambda_f(x_k) = \| \nabla f(x_k) \|_{x_k}$
- Local distance to the minimizer: $\| x_k - x^* \|_{x_k}$
- Distance to the minimizer under fixed metric $\| x_k - x^* \|_{x^*}$

Indeed, all of these measures are equivalent locally.

**Proposition 20.1** When $\lambda_f(x) < 1$, we have

1. $f(x) - f(x^*) \leq \omega_\ast(\lambda_f(x)) \leq \frac{\lambda_f(x)^2}{2(1 - \lambda_f(x))^2}$
2. $\| x - x^* \|_x \leq \frac{\lambda_f(x)}{1 - \lambda_f(x)}$
3. $\| x - x^* \|_{x^*} \leq \frac{\lambda_f(x)}{1 - \lambda_f(x)}$

**Proof:** See Theorem 4.1.13 in (Nesterov, 2004).

For this reason, we will focus mainly on the convergence in terms of $\lambda_f(x)$.

For simplicity, in the following, let us denote $\lambda_k := \lambda_f(x_k), k = 0, 1, 2, \ldots$

**Theorem 20.2 (Local convergence)** If $x_k \in \text{dom}(f)$ and $\lambda_k < 1$, then $x_{k+1} \in \text{dom}(f)$ and

$$\lambda_{k+1} \leq \left( \frac{\lambda_k}{1 - \lambda_k} \right)^2$$

**Proof:** Note $\| x_{k+1} - x_k \|_{x_k} = \lambda_f(x_k) = \lambda_k < 1$, so $x_{k+1} \in \text{dom}(f)$. Also, it holds that

$$\nabla^2 f(x_{k+1}) \succ (1 - \lambda_k)^2 \nabla^2 f(x_k)$$

Hence,

$$\lambda_{k+1} = \sqrt{\nabla f(x_{k+1})^T \left[ \nabla^2 f(x_{k+1}) \right]^{-1} \nabla f(x_{k+1})} \leq \frac{1}{1 - \lambda_k} \sqrt{\nabla f(x_{k+1})^T \left[ \nabla^2 f(x_k) \right]^{-1} \nabla f(x_{k+1})}$$
Note
\[ \nabla f(x_{k+1}) = \nabla f(x_{k+1}) - \nabla f(x_k) - [\nabla^2 f(x_k)](x_{k+1} - x_k) \\
= \left[ \int_0^1 \nabla^2 f(x_k + t(x_{k+1} - x_k)) - \nabla^2 f(x_k) dt \right](x_{k+1} - x_k) \\
: = G(x_{k+1} - x_k) \]

Hence,
\[ \lambda_{k+1} \leq \frac{1}{1 - \lambda_k} \sqrt{(x_{k+1} - x_k)^T G^T [\nabla^2 f(x_k)]^{-1} G(x_{k+1} - x_k)} \]
\[ \leq \frac{1}{1 - \lambda_k} \|x_{k+1} - x_k\|_{x_k} \cdot \| [\nabla^2 f(x_k)]^{-1/2} G [\nabla^2 f(x_k)]^{-1/2} \|_2 \]
\[ \leq \frac{\lambda_k}{1 - \lambda_k} \| [\nabla^2 f(x_k)]^{-1/2} G [\nabla^2 f(x_k)]^{-1/2} \|_2 \]

We have
\[ G \succeq \nabla^2 f(x_k) \int_0^1 \left[ (1 - t\lambda_k)^2 - 1 \right] dt = \left( \frac{\lambda^2_k}{3} \right) \nabla^2 f(x_k) \]
\[ G \preceq \nabla^2 f(x_k) \int_0^1 \left[ \frac{1}{(1 - \lambda_k)^2} - 1 \right] dt = \frac{\lambda_k}{1 - \lambda_k} \nabla^2 f(x_k) \]

Hence, \( \| H \|_2 \leq \max \left\{ \lambda - \frac{\lambda^2_k}{3}, \frac{\lambda_k}{1 - \lambda_k} \right\} = \frac{\lambda_k}{1 - \lambda_k} \)

This leads to the conclusion \( \lambda_{k+1} \leq \left( \frac{\lambda_k}{1 - \lambda_k} \right)^2 \)

Remark: Let \( \lambda^* \) be such that \( \frac{\lambda^*}{(1 - \lambda^*)^2} = 1 \). Then if \( \lambda_k < \lambda^* \), \( \lambda_{k+1} < \lambda_k \). The region of quadratic
convergence is \( \lambda_f(x) \leq \lambda^* = \frac{3 - \sqrt{5}}{2} \approx 0.38 \).

Question: The Newton method for self-concordant function still might diverge if not started with
a point with \( \lambda_f(x) \) small enough. How to modify the Newton method to ensure global convergence?

The remedy is to use Newton method with line-search or damping factors
\[ x_{k+1} = x_k - \gamma_k [\nabla^2 f(x_k)]^{-1} \nabla f(x_k) \]
where \( \gamma_k > 0 \) is some stepsize.

20.3 Damped Newton Method

Damped Newton method: initialize \( x_0 \in \text{dom}(f) \) and update via
\[ x_{k+1} = x_k - \frac{1}{1 + \lambda_f(x_k)} [\nabla^2 f(x_k)]^{-1} \nabla f(x_k) \]

Note this is essentially Newton method with particular steps size \( \gamma_k = \frac{1}{1 + \lambda_f(x_k)} \).
Remark.

1. Since $\|x_{k+1} - x_k\|_{x_k} = \frac{\lambda_f(x_k)}{1 + \lambda_f(x_k)} < 1$, $x_{k+1} \in W_1^{0}(x_k) \subseteq dom(f)$. The damped Newton procedure is always well-defined.

2. The Newton direction $d_k = -[\nabla^2 f(x_k)]^{-1}\nabla f(x_k)$ is a descent direction when $f$ is non-degenerate. If $f$ is bounded below, then $x_t$ converges to the unique minimizer of $f$.

Theorem 20.3 (Global convergence of damped Newton) The damped Newton method satisfies that

1. (Descent phase) $\forall k \geq 0, f(x_{k+1}) \leq f(x_k) - \omega(\lambda_f(x_k))$

2. (Quadratic convergence phase) If $\lambda_f(x_k) < \frac{1}{4}$, then $\lambda_f(x_{k+1}) \leq 2[\lambda_f(x_k)]^2$

Proof:

1. In view of previous theorem,

$$f(x_{k+1}) \leq f(x_k) + \langle\nabla f(x_k), x_{k+1} - x_k\rangle + \omega_s(\|x_{k+1} - x_k\|_{x_k})$$

$$= f(x_k) - \frac{\lambda_f(x_k)}{1 + \lambda_f(x_k)} + \omega_s\left(\frac{\lambda_f(x_k)}{1 + \lambda_f(x_k)}\right)$$

where $\omega_s(t) = -t - \ln(1 - t)$.

This can be further simplified as $f(x_{k+1}) \leq f(x_k) - \omega(\lambda_f(x_k))$.

2. Proof follows similarly as the earlier theorem for basic Newton method.

Remark: A general strategy for solving the self-concordant minimization:

- Damped Newton stage: when $\lambda_f(x_k) \geq \beta$, where $\beta \in (0, 1/4)$

  $$f(x_{k+1}) \leq f(x_k) - \omega(\beta)$$

  The number of steps of this stage is bounded by $N_1 \leq \frac{f(x_0) - f(x^*)}{\omega(\beta)}$.

- (Damped/Basic) Newton stage: when $\lambda_f(x_k) < \beta$

  $$\lambda_f(x_{k+1}) \leq 2[\lambda_f(x_k)]^2$$

  The number of steps to find a solution with $\lambda_f(x) \leq \epsilon$ is bounded by $N_2 \leq O(1) \log_2 \log_2\left(\frac{1}{\epsilon}\right)$.

The total complexity does not exceed

$$O(1)[f(x_0) - f^* + \log \log(\frac{1}{\epsilon})]$$
In this lecture, we cover the following topics

- Revisit barrier method
- Self-concordant barrier
- Restate path-following scheme

Reference: Nesterov, Introductory Lectures on Convex Optimization, 2004, Chapter 4.2

21.1 Revisit Barrier Method

Recall the barrier method solves the general convex problem

$$\min_{x \in X} f(x), \quad \text{where } X = \{x : g_i(x) \leq 0, i = 1, ..., m\}$$

by solving a sequence of unconstrained minimization: $\min_x tf(x) + F(x)$, where $F(x)$ is barrier function defined on $\text{int}(X)$, and $t > 0$ is penalty parameter.

Without loss of generality, let us consider problem of the form,

$$\min_{x \in X} c^T x$$

where $X$ is a closed bounded convex set with non-empty interior. Let $F$ be standard self-concordant with $\text{cl}(\text{dom}(F)) = X$. We want to solve (P) by tracing the central path

$$x^*(t) = \arg \min_x \underbrace{tc^T x + F(x)}_{F_t(x)}$$

By optimality condition: $\forall t > 0$,

$$tc + \nabla F(x^*(t)) = 0$$

Note that $F_t(x)$ is standard self-concordant and (damped) Newton method achieves a local convergence when $\lambda_{F_t(x)} \leq \frac{1}{4}$. 
When increasing $t \to t'$, we would want to preserve $\lambda F_t^*(x^*(t)) \leq \frac{1}{4}$ and make $t'$ as large as possible.

We have

$$\lambda F_t^*(x^*(t)) = \| \nabla F_t^*(x^*(t)) \|_{x^*(t),*}$$

$$= \| (t' - t)c + tc + \nabla F(x^*(t)) \|_{x^*(t),*}$$

$$= \| (t' - t)c \|_{x^*(t),*}$$

$$= \| \left( \frac{t'}{t} - 1 \right) \nabla F(x^*(t)) \|_{x^*(t),*}$$

Hence,

$$\frac{t'}{t} = 1 + \frac{1}{4\lambda F(x^*(t))}$$

To ensure $t \to +\infty$, need $\lambda F(x)$ to be uniformly bounded from above, namely,

$$\lambda F^2(x) = \nabla F(x)^T \left[ \nabla^2 F(x) \right]^{-1} \nabla F(x) \leq \nu$$

for some $\nu$. This leads to the definition of self-concordant barriers.

### 21.2 Self-concordant Barriers

#### 21.2.1 Definition and Examples

**Definition 21.1** Let $v \geq 0$. We call $F$ a $v$-self-concordant barrier (v-s.c.b.) for set $X = \text{cl}(\text{dom}(F))$ if $F$ is standard self-concordant and satisfies

$$|DF(x)[h]| \leq v^{1/2} \sqrt{D^2 F(x)[h,h]} \quad \forall x \in \text{dom}(F), h \in \mathbb{R}^n \quad (\star)$$

**Remark**

1. The inequality $(\star)$ implies that $|\nabla F(x)^T h| \leq \nu \| h \|_{x}^2$, i.e. $F$ is Lipschitz continuous, w.r.t. the local norm defined by $F$.

2. When $F$ is non-degenerate, $(\star)$ is equivalent to

$$\lambda F^2(x) = \nabla F(x)^T \left[ \nabla^2 F(x) \right]^{-1} \nabla F(x) \leq \nu$$

3. The following are equivalent:

$$(\star) \iff \nabla^2 F(x) \succ \frac{1}{\nu} \nabla F(x) \left[ \nabla F(x) \right]^T$$

**Examples**

- Constant function: $f(x) = c$ is 0-s.c.b.
• Linear function: \( f(x) = a^T x + c(a \neq 0) \), is not s.c.b.

• Quadratic function: \( f(x) = \frac{1}{2} x^T Q x + q^T x + c \) \( Q > 0 \) is not s.c.b.

• Logarithmic function: \( f(x) = -\ln x \) \((x > 0)\) is 1-s.c.b.
\[
\frac{(f'(x))^2}{f''(x)} = \left( -\frac{1}{x} \right)^2 x^2 = 1
\]

### 21.2.2 Self-concordance Preserving Operators

**Proposition 21.2** The following are true:

1. If \( F(x) \) is a \( \nu \)-self-concordant barrier, then \( \tilde{F}(y) = F(Ay + b) \) is a \( \nu \)-self-concordant barrier.

2. If \( F_i(x) \) is \( \nu_i \)-self-concordant barrier, \( i = 1, 2 \), then \( F_1(x) + F_2(x) \) is \((\nu_1 + \nu_2)\)-self-concordant barrier.

3. If \( F(x) \) is a \( \nu \)-self-concordant barrier, then \( \beta F(x) \) with \( \beta \geq 1 \) is a \((\beta \nu)\)-self-concordant barrier.

**Example:** The function
\[
F(x) = -\sum_{i=1}^{m} \ln(b_i - a_i^T x)
\]
is \( m \)-self-concordant barrier for the set \( \{ x : Ax \leq b \} \)

**Remark:** Indeed, for any closed convex set \( X \subseteq \mathbb{R}^n \) with non-empty interior, there exists a \((\beta n)\)-self-concordant barrier for \( X \).

### 21.2.3 Properties of Self-concordant Barriers

**Lemma 21.3** (Boundedness) Let \( f \) be a \( \nu \)-self-concordant barrier for \( X \). Then for any \( x \in \text{int}(X), y \in X \), we have \( \langle \nabla F(x), y - x \rangle \leq \nu \)

Proof is omitted and can be found in Theorem 4.2.4. in (Nesterov, 2004).

**Theorem 21.4** For any \( t > 0 \), we have
\[
c^T x^*(t) - \min_{x \in X} c^T x \leq \frac{\nu}{t}
\]

**Proof:**
\[
c^T x^*(t) - c^T y = -t^{-1} \nabla F(x^*(t))^T (x^*(t) - y) \leq \frac{\nu}{t}
\]
21.3 Restate Path-following Scheme

We would like to address the convex problem

\[ \min_{x \in X} c^T x \]

by tracing the central path

\[ x^*(t) = \arg \min_x \{ F_t(x) := tc^T x + F(x) \} \]

where \( F(x) \) is a \( \nu \)-self-concordant barrier for \( X = \text{cl}(\text{dom}(F)) \).

**Theorem 21.5** For any \( t > 0 \), we have

\[ c^T x^*(t) - \min_{x \in X} c^T x \leq \frac{\nu}{t} \]

**Proof:** By optimality condition: \( tc + \nabla F(x^*(t)) = 0 \), i.e. \( c = -t^{-1} \nabla F(x^*(t)) \),

\[ \forall y \in X : \quad c^T x^*(t) - c^T y = -t^{-1} \nabla F(x^*(t))^T (x^*(t) - y) = t^{-1} \nabla F(x^*(t))^T (y - x^*(t)) \leq \frac{\nu}{t} \]

Now consider an approximate solution \( x \) that is close to \( x^*(t) \):

\[ \lambda_{F_t}(x) \leq \beta \]

where \( \beta \) is small enough.

**Theorem 21.6** If \( \lambda_{F_t}(x) \leq \beta \),

\[ c^T x - \min_{x \in X} c^T x \leq \frac{1}{t} \left( \nu + \frac{\sqrt{\nu} \beta}{1 - \beta} \right) \]

**Proof:** First of all,

\[ c^T x - c^T x^*(t) \leq \| c \|_{x^*(t),*} \| x - x^*(t) \|_{x^*(t)} = t^{-1} \| \nabla F(x^*(t)) \|_{x^*(t),*} \| x - x^*(t) \|_{x^*(t)} \]

Recall from last lecture that for \( x^* = \arg \min_x f(x) \) with standard self concordant \( f \):

\[ \| x - x^* \|_{x^*} \leq \frac{\lambda_f(x)}{1 - \lambda_f(x)} \]
Hence,
\[ \| x - x^*(t) \|_{x^*(t)} \leq \frac{\lambda_F(x)}{1 - \lambda_F(x)} \leq \frac{\beta}{1 - \beta} \]

Thus,
\[ c^T x - c^T x^*(t) \leq \frac{\sqrt{\nu}}{t} \frac{\beta}{1 - \beta} \]

\[ c^T x - \min_{x \in X} c^T x = c^T x - c^T x^*(t) + c^T x^*(t) - \min_{x \in X} c^T x \]

\[ \leq \frac{\sqrt{\nu}}{t} \frac{\beta}{1 - \beta} + \frac{\nu}{t} \]

Now we can formally describe the path-following scheme.

**Path-following Scheme**

- Initialize \((x_0, t_0)\) with \(t_0 > 0\) and \(\lambda_{F_0}(x_0) \leq \beta \ (\beta \in (0, \frac{1}{4}))\)
- For \(k \geq 0\), do
  \[ t_{k+1} = t_k (1 + \frac{\gamma}{\sqrt{\nu}}) \]
  \[ x_{k+1} = x_k - [\nabla^2 F(x_k)]^{-1} [t_{k+1} c + \nabla F(x_k)] \]

**Theorem 21.7 (Rate of convergence)** In the above scheme, one has
\[ c^T x_k - \min_{x \in X} c^T x \leq O(1) \frac{\nu}{t_0} \exp \left\{ -O(1) \frac{k}{\sqrt{\nu}} \right\} \]
where the constant factor \(O(1)\) depends solely on \(\beta\) and \(\gamma\).

**Remark [Complexity]**

- The total complexity of Newton steps does not exceed
\[ N(\epsilon) \leq O \left( \sqrt{\nu} \log \frac{\nu}{t_0 \epsilon} \right) \]
- The total arithmetic cost of finding an \(\epsilon\)-solution by the above scheme does not exceed
\[ O \left( M \sqrt{\nu} \log \frac{\nu}{t_0 \epsilon} \right) \]
where \(M\) is the arithmetic cost for computing \(\nabla F(x)\), \(\nabla^2 F(x)\) and solving a Newton system.
Remark [Initialization]  In order to obtain a fair \((x_0, t_0)\), s.t.

\[
\lambda_{F_t}(x_0) \leq \beta
\]

one can apply the following trick. Suppose \(\hat{x} \in \text{dom}(F)\) is given. Consider the auxiliary path

\[
y^*(t) = \arg\min_x [-t\nabla F(\hat{x})^T x + F(x)]
\]

When \(t = 1\), \(y^*(1) = \hat{x}\). When \(t \to 0\), \(y^*(t) = x_F := \arg\min_x F(x)\).

We can trace \(y^*(t)\) as \(t\) decreases from 1 to 0, until we approach to a point \((y_0, t_0)\) such that

\[
\lambda_{F_t}(x_0) \leq \beta.
\]
In this lecture, we cover the following topics:

- IPM for Conic Programs
- Primal Dual path following IPM

### 22.1 Summary of IPM

Let us first summarize the key concepts discussed in the last few lectures.

- **Self-concordant barrier:** A function $F(x)$ is $\nu$-s.c.b. for a set $X = \text{cl}(\text{dom}(F))$ if it satisfies for any $x \in \text{dom}(F)$, $h \in \mathbb{R}^n$:

  $|D^3F(x)[h, h, h]| \leq 2(D^2F(x)[h, h])^{3/2}$  
  $|DF(x)[h]| \leq \sqrt{\nu}(D^2F(x)[h, h])^{1/2}$

- **Path-following interior point method:**

  $\min \ c^T x \quad \Longrightarrow \quad \min \ x \ t c^T x + F(x)$
  \[ \text{s.t.} \quad x \in X \]

1. choose $(x_0, t_0)$, where $t_0 > 0$ and $x_0$ is close to the analytical center $x_F := \arg \min_x F(x)$
2. do for $k = 0, 1, \ldots$

  $t_{k+1} = t_k (1 + \frac{\gamma}{\sqrt{\nu}})$
  $x_{k+1} = x_k - [\nabla^2 F(x_k)]^{-1}[t_{k+1}c + \nabla F(x_k)]$

- **Iteration Complexity:** $O(\sqrt{\nu \log(\frac{\nu}{\epsilon})})$ iterations.
22.2 IPM for Conic Programs

Recall the standard form and dual of conic program:

\[
\begin{align*}
\min_x & \quad c^T x \\
\text{s.t.} & \quad Ax - b \in K \\
\max_y & \quad b^T y \\
\text{s.t.} & \quad A^T y = c \\
y & \in K^* 
\end{align*}
\]

This becomes an LP, SOCP, SDP when \(K = \mathbb{R}_+^n, L^n, S^n_+\), respectively.

22.2.1 Self-concordant Barriers for LP, SOCP, SDP

We now introduce the canonical self-concordant barriers for these cones.

**Example 1:** \(F(x) = -\sum_{i=1}^{n} \ln(x_i)\) is \(n\)-s.c.b for the nonnegative orthant \(\mathbb{R}_+^n\).

**Example 2:** \(F(x) = -\ln(x_1^2 - x_2^2 - ... - x_{n-1}^2)\) is 2-s.c.b. for the Lorentz cone \(L^n\).

**Example 3:** \(F(X) = -\ln(\det(X)) = -\sum_{i=1}^{n} \ln(\lambda_i(X))\) is \(n\)-s.c.b. for the positive semidefinite cone \(S^n_+\).

The first two examples can be easily verified based on the definition. Let’s prove the third case.

**Proof:** It suffices to show that given \(X \in \text{int}(S^n_+)\) and \(H \in S^n\), the one-dimensional restriction

\[
\phi(t) = F(X + tH) = -\ln(\det(X + tH))
\]

is a \(n\)-self-concordant barrier. We have

\[
\begin{align*}
\phi(t) &= -\ln(\det(X^{1/2}(I + tX^{-1/2}HX^{-1/2})X^{1/2})) \\
&= -\ln(\det(I + tX^{-1/2}HX^{-1/2})) - \ln(\det(X)) \\
&= -\sum_{i=1}^{n} \ln(1 + t\lambda_i(X^{-1/2}HX^{-1/2})) + \phi(0)
\end{align*}
\]

Denote \(\lambda_i = \lambda_i(X^{-1/2}HX^{-1/2}), i = 1, ..., n\), then \(\phi(t) = -\sum_{i=1}^{n} \ln(1 + t\lambda_i) + \phi(0)\). Hence,

\[
\begin{align*}
\phi'(0) &= -\sum_{i=1}^{n} \lambda_i \\
\phi''(0) &= \sum_{i=1}^{n} \lambda_i^2 \\
\phi'''(0) &= -\sum_{i=1}^{n} \lambda_i^3
\end{align*}
\]
Note that
\[
\left( \sum_{i=1}^{n} \lambda_i \right)^2 \leq n \sum_{i=1}^{n} \lambda_i^2 \implies |\phi'(0)|^2 \leq n \phi''(0)
\]
\[
\left| \sum_{i=1}^{n} \lambda_i^3 \right| \leq \left( \sum_{i=1}^{n} \lambda_i^2 \right)^{3/2} \implies |\phi'''(0)| \leq 2[\phi''(0)]^{3/2}
\]
Therefore, \( \phi(t) \) is a \( n \)-self-concordant barrier for any \( X \in \text{int}(S^n_+ \) and \( H \in S^n \).

\[\text{Remark:} \text{ Indeed, any } \nu \text{-self-concordant carrier for the cone } S^n_+ \text{ has } \nu \geq n. \]

As a result, the function build on the above barriers,
\[
\tilde{F}(x) := F(Ax - b)
\]
is a self-concordant barrier for \( X := \{x : Ax - b \in K\} \).

### 22.2.2 Linear Program: IPM vs Ellipsoid Method

For instance, consider the linear program \((m > n)\)
\[
\begin{align*}
\min & \quad c^T x \\
\text{s.t.} & \quad a_j^T x \geq b_j, \quad j = 1, ..., m
\end{align*}
\]
The barrier function
\[
\tilde{F}(x) = -\sum_{j=1}^{m} \ln(a_j^T x - b_j)
\]
is \( m \)-self-concordant barrier for the constraint set.

We can compute the gradient and Hessian
\[
\nabla \tilde{F}(x) = -\sum_{j=1}^{m} \frac{a_j}{a_j^T x - b_j}
\]
\[
\nabla^2 \tilde{F}(x) = \sum_{j=1}^{m} \frac{a_j a_j^T}{(a_j^T x - b_j)^2}
\]
The arithmetic costs of computing \( \tilde{F}(x), \nabla \tilde{F}(x), \nabla^2 \tilde{F}(x) \) are \( O(mn), O(mn), O(mn^2) \), respectively.

The arithmetic cost for a Newton step, i.e., solving \( [\nabla^2 \tilde{F}(x)] y = \nabla \tilde{F}(x) \), is \( O(n^3) \). Hence, the overall arithmetic cost of finding a \( \epsilon \)-solution to the linear program is
\[
O(mn^2)O(\sqrt{m} \log(m \epsilon)) = O(m^{3/2}n^2 \log(m \epsilon))
\]
In contrast, when applying Ellipsoid method, the overall arithmetic cost is
\[
O(mn + n^2)O(n^2 \log(1 \epsilon)) = O(mn^3 \log(1 \epsilon))
\]
The interior-point method is more efficient when \( m \) is not too large, i.e. \( m \leq O(n^2) \).
22.3 Primal-Dual Path-Following IPM

When solving conic programs, ideally we would like to develop interior point methods that

- produce primal-dual pairs at each iteration
- handle equality constraints
- require no prior knowledge of a strictly feasible solution
- adjust penalty based on current solution

**Key idea:** To approximate the KKT conditions.

We focus on the SDP case. Consider the standard form of the primal problem

\[
\begin{align*}
\text{min} & \quad \text{Tr}(CX) \\
\text{s.t.} & \quad \text{Tr}(A_i X) = b_i, \ i = 1, ..., m \\
& \quad X \succ 0 
\end{align*}
\]

and its dual problem

\[
\begin{align*}
\text{max} \quad & \quad b^T y \\
\text{s.t.} & \quad \sum_{i=1}^{m} y_i A_i + Z = C \\
& \quad Z \succ 0 
\end{align*}
\]

Assume \((P)\) and \((D)\) are strictly primal-dual feasible, so there is no duality gap.

**KKT conditions for \((P)\) and \((D)\):**

\[
\begin{align*}
X^* \succ 0, Z^* \succ 0 \\
\forall i = 1, ..., m : \text{Tr}(A_i X^*) = b_i \\
\sum_{i=1}^{m} y_i^* A_i + Z^* = C \\
X^* Z^* = 0
\end{align*}
\]

(primal-dual feasibility) (complementary slackness)
We now consider the barrier problem of the primal problem

\[
\min \quad \text{Tr}(CX) - \mu \ln(\det(X)) \\
\text{s.t.} \quad \text{Tr}(A_i X) = b_i, \quad i = 1, \ldots, m
\]

(BP)

and that of the dual problem

\[
\max_y \quad b^T y + \mu \log(\det(Z)) \\
\text{s.t.} \quad \sum_{i=1}^{m} y_i A_i + Z = C
\]

(BD)

In fact, these are indeed the Lagrange duals to each other, up to constant.

**KKT conditions for (BP) and (BD):**

\[
\forall i = 1, \ldots, m : \text{Tr}(A_i X^*(\mu)) = b_i \\
\sum_{i=1}^{m} y_i^*(\mu) A_i + Z^*(\mu) = C
\]

(primal-dual feasibility)

\[
X^*(\mu)Z^*(\mu) = \mu I
\]

(complementary slackness)

The duality gap at \((X^*(\mu), y^*(\mu))\) is

\[
\text{Tr}(CX^*(\mu)) - b^T y^*(\mu) = \text{Tr}(Z^*(\mu)X^*(\mu)) = \mu n
\]

As \(\mu \to 0\), the duality gap is zero, and \((X^*(\mu), y^*(\mu), Z^*(\mu)) \to (X^*, y^*, Z^*)\).

The set \(\{(X(\mu), y(\mu), Z(\mu)) : \mu > 0\}\) is called the **primal-dual central path**.

**Newton step:** Find direction \((\Delta X, \Delta y, \Delta Z)\) by solving the equations:

\[
\begin{align*}
\text{Tr}(A_i(X + \Delta X)) = b_i, \quad i = 1, \ldots, m \\
\sum_{i=1}^{m} (y_i + \Delta y_i) A_i + (Z + \Delta Z) = C \\
(X + \Delta X)(Z + \Delta Z) = \mu I \\
\text{Tr}(A_i \Delta X) = 0, \quad i = 1, \ldots, m \\
\sum_{i=1}^{m} \Delta y_i A_i + \Delta Z = 0 \\
(X + \Delta X)(Z + \Delta Z) = \mu I
\end{align*}
\]
Basic primal-dual scheme

Initial \((X, y, Z) = (X_0, y_0, Z_0)\) with \(X_0 > 0, Z_0 > 0\).

At each iteration:

- compute \(\mu = \frac{\text{Tr}(XZ)}{n}\), \(\mu \leftarrow \frac{\mu}{2}\)
- compute \((\Delta X, \Delta y, \Delta Z)\) by solving the KKT equations (*)
- update \((X, y, Z) \leftarrow (X + \alpha \Delta X, y + \beta \Delta y, Z + \beta \Delta Z)\) with proper \(\alpha, \beta\) that preserves positivity of \((X, Z)\).

Remark (Approximation of KKT equation). Note that only the last equation in the system of KKT conditions is nonlinear. One can apply first-order approximation:

\[
\mu = (X + \Delta X)(Z + \Delta Z) \approx XZ + \Delta XZ + X\Delta Z
\]

and solve the linearized KKT equations.
Lecture 23: CVX Tutorial

April 19, 2017

(Slides Origin: Boyd & Vandenberghe)
Cone program solvers

- **LP solvers**
  - many, open source and commercial

- **cone solvers**
  - each handles combinations of a subset of LP, SOCP, SDP, EXP cones
  - open source: SDPT3, SeDuMi, CVXOPT, CSDP, ECOS, SCS, . . .
  - commercial: Mosek, Gurobi, Cplex, . . .
Transforming problems to cone form

• lots of tricks for transforming a problem into an equivalent cone program
  – introducing slack variables
  – introducing new variables that upper bound expressions

• these tricks greatly extend the applicability of cone solvers

• writing code to carry out this transformation is painful

• modeling systems automate this step
Modeling systems

a typical modeling system

- automates transformation to cone form; supports
  - declaring optimization variables
  - describing the objective function
  - describing the constraints
  - choosing (and configuring) the solver

- when given a problem instance, calls the solver

- interprets and returns the solver’s status (optimal, infeasible, . . . )

- (when solved) transforms the solution back to original form
Some current modeling systems

- AMPL & GAMS (proprietary)
  - developed in the 1980s, still widely used in traditional OR
  - no support for convex optimization
- YALMIP (‘Yet Another LMI Parser’, matlab)
  - first object-oriented convex optimization modeling system
- CVX (matlab)
- CVXPY (python, GPL)
- Convex.jl (Julia, GPL, merging into JUMP)
- CVX, CVXPY, and Convex.jl collectively referred to as CVX*
Disciplined convex programming

- describe objective and constraints using expressions formed from
  - a set of basic atoms (affine, convex, concave functions)
  - a restricted set of operations or rules (that preserve convexity)

- modeling system keeps track of affine, convex, concave expressions

- rules ensure that
  - expressions recognized as convex are convex
  - but, some convex expressions are not recognized as convex

- problems described using DCP are convex by construction

- all convex optimization modeling systems use DCP
CVX

- uses DCP
- runs in Matlab, between the `cvx_begin` and `cvx_end` commands
- relies on SDPT3 or SeDuMi (LP/SOCP/SDP) solvers
- refer to user guide, online help for more info
- the CVX example library has more than a hundred examples
Example: Constrained norm minimization

```matlab
A = randn(5, 3);
b = randn(5, 1);
cvx_begin
    variable x(3);
    minimize(norm(A*x - b, 1))
    subject to
        -0.5 <= x;
        x <= 0.3;
end
```

- between `cvx_begin` and `cvx_end`, `x` is a CVX variable
- statement `subject to` does nothing, but can be added for readability
- inequalities are interpreted elementwise
What CVX does

after cvx_end, CVX

• transforms problem into an LP
• calls solver SDPT3
• overwrites (object) x with (numeric) optimal value
• assigns problem optimal value to cvx_optval
• assigns problem status (which here is Solved) to cvx_status

(had problem been infeasible, cvx_status would be Infeasible and x would be NaN)
• declare variables with variable name[(dims)] [attributes]
  – variable x(3);
  – variable C(4,3);
  – variable S(3,3) symmetric;
  – variable D(3,3) diagonal;
  – variables y z;
Affine expressions

- form affine expressions

\[
A = \text{randn}(4, 3);
\]
\[
\text{variables } x(3) \ y(4);
\]
- $3x + 4$
- $A \cdot x - y$
- $x(2:3)$
- $\text{sum}(x)$
### Some functions

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<th>attributes</th>
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<td>cvx</td>
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<tr>
<td>huber(x)</td>
<td>$\begin{cases} x^2, &amp;</td>
<td>x</td>
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</tbody>
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Composition rules

• can combine atoms using valid composition rules, e.g.:
  
  – a convex function of an affine function is convex
  – the negative of a convex function is concave
  – a convex, nondecreasing function of a convex function is convex
  – a concave, nondecreasing function of a concave function function is concave

• for convex \( h, h(g_1, \ldots, g_k) \) is recognized as convex if, for each \( i \),
  
  – \( g_i \) is affine, or
  – \( g_i \) is convex and \( h \) is nondecreasing in its \( i \)th arg, or
  – \( g_i \) is concave and \( h \) is nonincreasing in its \( i \)th arg

• for concave \( h, h(g_1, \ldots, g_k) \) is recognized as concave if, for each \( i \),
  
  – \( g_i \) is affine, or
  – \( g_i \) is convex and \( h \) is nonincreasing in \( i \)th arg, or
  – \( g_i \) is concave and \( h \) is nondecreasing in \( i \)th arg
Valid (recognized) examples

u, v, x, y are scalar variables; X is a symmetric $3 \times 3$ variable

- convex:
  - $\text{norm}(A \cdot x - y) + 0.1 \cdot \text{norm}(x, 1)$
  - $\text{quad_over_lin}(u - v, 1 - \text{square}(v))$
  - $\text{lambda_max}(2 \cdot X - 4 \cdot \text{eye}(3))$
  - $\text{norm}(2 \cdot X - 3, \ 'fro')$

- concave:
  - $\text{min}(1 + 2 \cdot u, 1 - \text{max}(2, v))$
  - $\text{sqrt}(v) - 4.55 \cdot \text{inv_pos}(u - v)$
Rejected examples

u, v, x, y are scalar variables

• neither convex nor concave:
  – square(x) - square(y)
  – norm(A*x - y) - 0.1*norm(x, 1)

• rejected due to limited DCP ruleset:
  – sqrt(sum(square(x))) (is convex; could use norm(x))
  – square(1 + x^2) (is convex; could use square_pos(1 + x^2), or 1 + 2*pow_pos(x, 2) + pow_pos(x, 4))
Sets

- some constraints are more naturally expressed with convex sets

- sets in CVX work by creating unnamed variables constrained to the set

- examples:
  - semidefinite(n)
  - nonnegative(n)
  - simplex(n)
  - lorentz(n)

- semidefinite(n), say, returns an unnamed (symmetric matrix) variable that is constrained to be positive semidefinite
Using the semidefinite cone

variables: \( X \) (symmetric matrix), \( z \) (vector), \( t \) (scalar)
constants: \( A \) and \( B \) (matrices)

- \( X == \text{semidefinite}(n) \)
  - means \( X \in S^{n}_+ \) (or \( X \succeq 0 \))

- \( A*X*A' - X == B*\text{semidefinite}(n)*B' \)
  - means \( \exists Z \succeq 0 \) so that \( AXA^T - X = BZB^T \)

- \( [X \ z; \ z' \ t] == \text{semidefinite}(n+1) \)
  - means \( \begin{bmatrix} X & z \\ z^T & t \end{bmatrix} \succeq 0 \)
Objectives and constraints

- **objective** can be
  - minimize(convex expression)
  - maximize(concave expression)
  - omitted (feasibility problem)

- **constraints** can be
  - convex expression <= concave expression
  - concave expression >= convex expression
  - affine expression == affine expression
  - omitted (unconstrained problem)
More involved example

A = randn(5);
A = A'*A;
cvx_begin
    variable X(5, 5) symmetric;
    variable y;
    minimize(norm(X) - 10*sqrt(y))
    subject to
        X - A == semidefinite(5);
        X(2,5) == 2*y;
        X(3,1) >= 0.8;
        y <= 4;
$cvx_end$
Defining new functions

- can make a new function using existing atoms

- **example**: the convex deadzone function

\[
f(x) = \max\{|x| - 1, 0\} = \begin{cases} 
0, & |x| \leq 1 \\
x - 1, & x > 1 \\
1 - x, & x < -1 
\end{cases}
\]

- create a file `deadzone.m` with the code

```matlab
function y = deadzone(x)
    y = max(abs(x) - 1, 0)
end
```

- `deadzone` makes sense both within and outside of CVX
Defining functions via incompletely specified problems

• suppose $f_0, \ldots, f_m$ are convex in $(x, z)$

• let $\phi(x)$ be optimal value of convex problem, with variable $z$ and parameter $x$

\[
\begin{align*}
\text{minimize} & \quad f_0(x, z) \\
\text{subject to} & \quad f_i(x, z) \leq 0, \quad i = 1, \ldots, m \\
& \quad A_1x + A_2z = b
\end{align*}
\]

• $\phi$ is a convex function

• problem above sometimes called *incompletely specified* since $x$ isn’t (yet) given

• an incompletely specified concave maximization problem defines a concave function
CVX functions via incompletely specified problems

Implement in CVX with

```matlab
function cvx_optval = phi(x)
cvx_begin
    variable z;
    minimize(f0(x, z))
    subject to
        f1(x, z) <= 0; ...
        A1*x + A2*z == b;
cvx_end
```

- Function \( \phi \) will work for numeric \( x \) (by solving the problem)

- Function \( \phi \) can also be used inside a CVX specification, wherever a convex function can be used
Simple example: Two element max

- create file `max2.m` containing

```
function cvx_optval = max2(x, y)
    cvx_begin
    variable t;
    minimize(t)
    subject to
        x <= t;
        y <= t;
    cvx_end
```

- the constraints define the epigraph of the max function
- could add logic to return `max(x, y)` when `x, y` are numeric (otherwise, an LP is solved to evaluate the max of two numbers!)
A more complex example

- $f(x) = x + x^{1.5} + x^{2.5}$, with $\text{dom } f = \mathbb{R}_+$, is a convex, monotone increasing function.

- its inverse $g = f^{-1}$ is concave, monotone increasing, with $\text{dom } g = \mathbb{R}_+$

- there is no closed form expression for $g$

- $g(y)$ is optimal value of problem

$$
\begin{align*}
\text{maximize} & \quad t \\
\text{subject to} & \quad t_+ + t_+^{1.5} + t_+^{2.5} \leq y
\end{align*}
$$

(for $y < 0$, this problem is infeasible, so optimal value is $-\infty$)
• implement as
  
  function cvx_optval = g(y)
  cvx_begin
      variable t;
      maximize(t)
      subject to
              pos(t) + pow_pos(t, 1.5) + pow_pos(t, 2.5) <= y;
  cvx_end

• use it as an ordinary function, as in \( g(14.3) \), or within CVX as a concave function:
  
  cvx_begin
      variables x y;
      minimize(quad_over_lin(x, y) + 4*x + 5*y)
      subject to
              g(x) + 2*g(y) >= 2;
  cvx_end
Example

• optimal value of LP

\[ f(c) = \inf \{ c^T x \mid Ax \leq b \} \]

is concave function of \( c \)

• by duality (assuming feasibility of \( Ax \leq b \)) we have

\[ f(c) = \sup \{-\lambda^T b \mid A^T \lambda + c = 0, \lambda \geq 0\} \]
• define \( f \) in CVX as

```matlab
function cvx_optval = lp_opt_val(A,b,c)
    cvx_begin
        variable lambda(length(b));
        maximize(-lambda'*b);
        subject to
            A'*lambda + c == 0; lambda >= 0;
    cvx_end
```

• in `lp_opt_val(A,b,c)` A, b must be constant; c can be affine
CVX hints/warnings

- watch out for = (assignment) versus == (equality constraint)
- $X \geq 0$, with matrix $X$, is an elementwise inequality
- $X \geq \text{semidefinite}(n)$ means: $X$ is elementwise larger than some positive semidefinite matrix (which is likely not what you want)
- writing subject to is unnecessary (but can look nicer)
- many problems traditionally stated using convex quadratic forms can posed as norm problems (which can have better numerical properties): $x'Px \leq 1$ can be replaced with $\text{norm}(\text{chol}(P)x) \leq 1$
Useful Resources

24.1 Optimization Algorithms and Convergence Rates

Till now, we have discussed several important algorithms for solving convex optimization:

- Ellipsoid method: poly-time algorithm, black-box method, requires first-order and separation oracles
- Interior point method: poly-time algorithm, barrier method, requires structural assumptions on the domain and self-concordant barriers
- Newton method: (local) quadratic convergent algorithm, black-box method, requires smoothness assumptions on the objective and first order and second-order oracles

While the above algorithms have the capability to solve convex programs to high accuracy within a small number of iterations, they suffer from very expensive iteration cost (often cubically in terms of the problem size), which eventually become impractical for large-scale convex problems.

First-order Methods:

For large-scale convex optimization, simpler algorithms such as first-order methods essentially become the only method of the choices. There exists a dedicated library of efficient first-order optimization algorithms:

- Gradient descent
- Nesterov’s accelerated gradient descent and variants (FISTA, geometric descent, etc)
- Coordinate descent and many variants
- Conditional gradient (a.k.a. Frank-Wolfe method)
- Subgradient methods
- Primal-dual methods (Arrow-Hurwicz method, etc)
- Proximal and operator splitting methods (proximal gradient method, ADMM, etc)
- Stochastic and incremental gradient methods (stochastic gradient descent, SVRG, etc)
In the rest of the semester we will present several primary methods mainly for the generic non-differentiable constrained problems.

**Rate of Convergence:**
Suppose the sequence \( \{x_k\} \) converges to \( x^* \)

**Definition 24.1 (Q-convergence)** The convergence rate is said to be

- **linear**: if \( \exists q \in (0, 1) \), such that \( \limsup_{k \to +\infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} = q \)

- **sublinear**: if \( \limsup_{k \to +\infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} = 0 \)

- **superlinear**: if \( \limsup_{k \to +\infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} = 1 \)

- **quadratic**: if \( \limsup_{k \to +\infty} \frac{\|x_{k+1} - x^*\|^2}{\|x_k - x^*\|^2} < +\infty \)

These are also called Q-convergence (quotient).

**Example:** \( x_k = \frac{1}{2^k}, \frac{1}{2}, (\frac{1}{2})^{2k} \) are Q-linear, Q-sublinear, Q-superlinear convergence.

Note that Q-convergence can be troublesome, for example, consider the sequence

\[
x_k = \begin{cases} 1 + \frac{1}{2^k}, & \text{k is even} \\ 1, & \text{k is odd.} \end{cases}
\]

**Definition 24.2 (R-convergence)** We say \( \{x_k\} \) converge to \( x^* \) R-linearly if \( \exists \{\delta_k\} \) s.t. \( \|x_k - x^*\| \leq \delta_k \) and \( \{\delta_k\} \) converges Q-linearly to \( x^* \).

For simplicity, we will drop the ‘Q’ and ‘R’.

### 24.2 Subgradient Method

Consider the generic convex minimization

\[
\begin{align*}
\min & \quad f(x) \\
\text{s.t.} & \quad x \in X
\end{align*}
\]

where

- \( f \) is convex and possibly non-differentiable
- \( X \) is non-empty, closed and convex.
Assume the problem is solvable with optimal solution and value denoted as $x^*, f^*$. Recall that a vector $g \in \partial f(x)$ is called a subgradient of $f$ at $x$ if

$$f(y) \geq f(x) + g^T(y - x) \quad \forall y \in \text{dom}(f)$$

The subgradient inequality can be interpreted as a supporting hyperplane for the level set $L_{f(x)}(f) = \{y : f(y) \leq f(x)\}$:

$$g^T(y - x) \leq 0, \forall y \in L_{f(x)}(f) = \{y : f(y) \leq f(x)\}$$

### 24.2.1 Subgradient Method (N. Shor, 1967)

Subgradient method works as follows: start with $x_1 \in X$ and update

$$x_{t+1} = \Pi_X(x_t - \gamma_t g_t)$$

where

- $g_t \in \partial f(x_t)$ is a subgradient of $f$ at $x_t$.
- $\gamma_t > 0$ is a proper stepsize
- $\Pi_X(x) = \arg\min_{y \in X} \| y - x \|_2$ is the Euclidean projection.

**Remark:**

- When $X = \mathbb{R}^n$, $f$ is continuously differentiable, this reduces to

$$x_{t+1} = x_t - \alpha_t \nabla f(x_t)$$

which is the well-known Gradient Descent method

- Unlike Gradient Descent, the negative direction of subgradient is not always a descent direction.

**Choices of Stepsize:**

- Constant stepsize: $\gamma_t = \gamma$, $\forall t$
- Diminishing stepsize: $\gamma_t \to 0$ and $\sum_{t=1}^{\infty} \gamma_t = +\infty$, e.g. $\gamma_t = \frac{1}{\sqrt{t}}$
- Square summable stepsize: $\sum_{t=1}^{\infty} \gamma_t^2 < +\infty$ and $\sum_{t=1}^{\infty} \gamma_t = +\infty$, e.g. $\gamma_t = \frac{1}{t}$
- Scaled stepsize: $\gamma_t = \frac{\gamma}{\|g_t\|_2}$
- Dynamic stepsize (Polyak’s optimal stepsize): $\gamma_t = \frac{f(x_t) - f^*}{\|g_t\|_2^2}$
In this lecture, we cover the following topics

- Subgradient Method
- Bundle Method
  - Kelley cutting plane method
  - Level method

Reference: Nesterov 2004. Chapter 3.2.3, 3.3

25.1 Subgradient Method

Recall that subgradient method works as follows

\[ x_{t+1} = \Pi_X(x_t - \gamma_t g_t), \quad t = 1, 2, \ldots \]

where \( g_t \in \partial f(x_t) \), \( \gamma_t > 0 \) and \( \Pi_X(x) = \arg \min_{y \in X} \| y - x \|_2 \) is the projection operator.

Note that the projection on \( X \) is easy to compute when \( X \) is simple, e.g. \( X \) is a ball, box, simplex, polyhedron, etc.

Lemma 25.1 (Projection) \( \forall x \in \mathbb{R}^n, z \in X, \)

\[ \| x - z \|^2 \geq \| x - \Pi_X(x) \|^2 + \| z - \Pi_X(x) \|^2 \]

Proof: When \( x \in X \), the inequality immediately hold true. Let \( x \notin X \). By definition.

\[ \Pi_X(x) = \arg \min_{x \in X} \| z - x \|^2 \]

By optimality condition, this implies \( 2(\Pi_X(x) - x)^T(z - \Pi_X(x)) \geq 0, \forall z \in X \). Hence,

\[ \| x - z \|^2 \geq \| x - \Pi_X(x) + \Pi_X(x) - z \|^2 \]

\[ \geq \| x - \Pi_X(x) \|^2 + \| \Pi_X(x) - z \|^2 \]
Lemma 25.2 (Key relation) For the subgradient method, we have
\[ \| x_{t+1} - x^* \|_2^2 \geq \| x_t - x^* \|_2^2 - 2\gamma_t (f(x_t) - f^*) + \gamma_t^2 \| g_t \|_2^2 \] (*)

Proof:
\[
\| x_{t+1} - x^* \|_2^2 = \| \Pi_X(x_t - \gamma_t g_t) - x^* \|_2^2 \\
\leq \| x_t - \gamma_t g_t - x^* \|_2^2 \\
= \| x_t - x^* \|_2^2 - 2\gamma_t g_t^T (x_t - x^*) + \gamma_t^2 \| g_t \|_2^2
\]

Due to convexity of \( f \), we have \( f^* \geq f(x_t) + g_t^T (x^* - x_t) \), i.e.
\[
g_t^T (x_t - x^*) > f(x_t) - f^*
\]
Combing there two inequalities leads to the desired result.

Remark: Note that when \( f^* \) is known, we can choose the ‘optimal’ \( \gamma_t \) by minimizing the right hand side of (\(*)\): \( \gamma_t^* = \frac{f(x_t) - f^*}{\| g_t \|_2} \), which is the Polyak’s stepsize.

In fact, knowing \( f^* \) is not a problem sometimes. For instance, when the goal is to solve the convex feasibility problem: Find \( x^* \in X \), s.t. \( g_i(x) \leq 0 \), \( i = 1, ..., m \). We can formulate this as
\[
\min_{x \in X} \max_{1 \leq i \leq m} g_i(x) \text{ or } \min_{x \in X} \sum_{i=1}^{m} \max(g_i(x), 0)
\]
The optimal value \( f^* \) is known to be 0 in this case.

If \( f^* \) is not known, one can replace \( f^* \) by its online estimate.

Theorem 25.3 (Convergence) Suppose \( f(x) \) is convex and Lipschitz continuous on \( X \):
\[ |f(x) - f(y)| \leq M_f \| x - y \|_2, \quad \forall x, y \in X \]
where \( M_f < +\infty \). Then the subgradient method satisfies:
\[
f(\tilde{x}_T) - f^* \leq \frac{\| x_1 - x^* \|_2^2 + \sum_{t=1}^{T} \gamma_t^2 M_f^2}{2 \sum_{t=1}^{T} \gamma_t}
\]
where \( \tilde{x}_T = (\sum_{t=1}^{T} \gamma_t)^{-1}(\sum_{t=1}^{T} \gamma_t x_t) \)

Proof: The Lipschitz continuity implies that \( \| g_t \|_2 \leq M_f, \forall t \). Summing up the key relation (\*) from \( t = 1 \) to \( t = T \), we obtain
\[
2 \sum_{t=1}^{T} \gamma_t (f(x_t) - f^*) \leq \| x_1 - x^* \|_2^2 - \| x_{T+1} - x^* \|_2^2 + \sum_{t=1}^{T} \gamma_t^2 M_f^2
\]
By convexity of \( f \): \( \sum_{t=1}^{T} \gamma_t f(x_t) \leq \sum_{t=1}^{T} \gamma_t f(x_t) \) This further leads to
\[
\left( \sum_{t=1}^{T} \gamma_t \right) [f(\tilde{x}_T) - f^*] \leq \frac{1}{2} \| x_1 - x^* \|_2^2 + \frac{M_f^2}{2} \sum_{t=1}^{T} \gamma_t^2
\]
and concludes the proof.
Convergence under various stepsize  Assume \( D_X = \max_{x, y} \| x - y \|_2 \) is the diameter of the set \( X \). It is interesting to see how the bounds in the above theorem would imply the convergence and even the convergence rate with different choices of stepsizes. By abuse of notation, we denote both \( \min_{1 \leq t \leq T} f(x_t) - f_* \) and \( f(\hat{x}_T) - f_* \) as \( \epsilon_T \).

1. **Constant stepsize:** with \( \gamma_t \equiv \gamma \),
   \[
   \epsilon_T \leq \frac{D_X^2 + T \gamma^2 M_f^2}{2T \gamma} = \frac{D_X^2}{2T} \cdot \frac{1}{\gamma} + \frac{M_f^2}{2} \gamma \overset{T \to \infty}{\longrightarrow} \frac{M_f^2}{2} \gamma.
   \]

   It is worth noticing that the error upper-bound does not diminish to zero as \( T \) grows to infinity, which shows one of the drawbacks of using arbitrary constant stepsizes. In addition, to optimize the upper bound, we can select the optimal stepsize \( \gamma_* \) to obtain:
   \[
   \gamma_* = \frac{D_X}{M_f \sqrt{T}} \Rightarrow \epsilon_T \leq \frac{D_X M_f}{\sqrt{T}}.
   \]

   It is shown that under this optimal choice \( \epsilon_T \sim O(\frac{D_X M_f}{\sqrt{T}}) \). However, this exhibits another drawback of constant stepsize that in practice \( T \) is not known in prior for evaluating the optimal \( \gamma_* \).

2. **Scaled stepsize:** with \( \gamma_t = \frac{\gamma}{\| g(x_t) \|_2} \),
   \[
   \epsilon_T \leq \frac{D_X^2 + \gamma^2 T}{2\gamma \sum_{t=1}^{T} 1/\| g(x_t) \|_2} \leq M_f \left( \frac{\Omega}{2T} \cdot \frac{1}{\gamma} + \frac{1}{2} \frac{1}{\gamma} \right) \overset{T \to \infty}{\longrightarrow} \frac{M_f^2}{2} \gamma.
   \]

   Similarly, we can select the optimal \( \gamma \) by minimizing the right hand side, i.e. \( \gamma_* = \frac{D_X}{\sqrt{T}} \).
   \[
   \gamma_t = \frac{D_X}{\sqrt{T} \| g(x_t) \|_2} \Rightarrow \epsilon_T \leq \frac{D_X M_f}{\sqrt{T}}.
   \]

   The same convergence rate is achieved while the same drawback about not knowing \( T \) in prior still exists in choosing \( \gamma_t \).

3. **Non-summable but diminishing stepsize:**
   \[
   \epsilon_T \leq \left( D_X^2 + \sum_{t=1}^{T} \gamma_t^2 M_f^2 \right) / \left( 2 \sum_{t=1}^{T} \gamma_t \right)
   \]
   \[
   \leq \left( D_X^2 + \sum_{t=1}^{T_1} \gamma_t^2 M_f^2 \right) / \left( 2 \sum_{t=1}^{T_1} \gamma_t \right) + \left( M_f^2 \sum_{t=T_1+1}^{T} \gamma_t^2 \right) / \left( 2 \sum_{t=T_1+1}^{T} \gamma_t \right)
   \]

   where \( 1 \leq T_1 \leq T \). When \( T \to \infty \), select large \( T_1 \) and the first term on the right hand side \( \to 0 \) since \( \gamma_t \) is non-summable. The second term also \( \to 0 \) because \( \gamma_t^2 \) always approaches zero faster than \( \gamma_t \). Consequently, we know that
   \[
   \epsilon_T \overset{T \to \infty}{\longrightarrow} 0.
   \]
An example choice of the stepsize is \(\gamma_t = O\left(\frac{1}{t^q}\right)\) with \(q \in (0, 1]\). As in the above cases, if we choose \(\gamma_t = \frac{\sqrt{2\Omega}}{M\sqrt{t}}\), then

\[
\gamma_t = \frac{D_X}{M\sqrt{t}} \Rightarrow \epsilon T \leq O\left(\frac{D_X M_f \ln(T)}{\sqrt{T}}\right).
\]

In fact, if we choose the averaging from \(\frac{T}{2}\) instead of 1, we have

\[
\min_{T/2 \leq t \leq T} f(x_t) - f_* \leq O\left(\frac{M_f \cdot D_X}{\sqrt{T}}\right).
\]

4. **Non-summable but square-summable stepsize:** It is obvious that

\[
\epsilon_T \leq \left(\Omega + \frac{M^2}{2} \sum_{t=1}^{T} \gamma_t^2\right) / \left(\sum_{t=1}^{T} \gamma_t\right) \xrightarrow{T \to \infty} 0.
\]

A typical choice of \(\gamma_t = \frac{1}{t^{1+q}}, q > 0\) also result in the rate of \(O\left(\frac{1}{\sqrt{T}}\right)\).

5. **Polyak stepsize:** The stepsize yields

\[
\|x_{t+1} - x_*\|_2^2 \leq \|x_t - x_*\|_2^2 - \frac{(f(x_t) - f_*)^2}{\|g(x_t)\|_2^2},
\]

which guarantees \(\|x_t - x_*\|_2^2\) decreases each step. Applying (25.1) recursively, we obtain

\[
\sum_{t=1}^{T} (f(x_t) - f_*)^2 \leq D_X^2 \cdot M_f < \infty.
\]

Therefore we have \(\epsilon_T \to 0\) as \(T \to \infty\) and \(\epsilon_T \leq O\left(\frac{1}{\sqrt{T}}\right)\).

**Corollary 25.4** When \(T\) is known, setting \(\gamma_t \equiv \frac{D_X}{M\sqrt{T}}\), in particular, we have

\[
f(\hat{x}_T) - f^* \leq \frac{D_X M_f}{\sqrt{T}}
\]

**Remark:** Subgradient method converges sublinearly. For an accuracy \(\epsilon > 0\), need \(O\left(\frac{D_X^2 M_f^2}{\epsilon^2}\right)\) number of iterations.

### 25.2 Bundle Method

When running the subgradient method, we obtain a bundle of affine underestimate of \(f(x)\):

\[
f(x_t) + g_t^T (x - x_t), \quad t = 1, 2, ...
\]
Definition 25.5 The piecewise linear function:

\[ f_t(x) = \max_{1 \leq i \leq t} \{ f(x_i) + g_i^T (x - x_i) \} \]

where \( g_i \in \partial f(x_i) \) is called the \( t \)-th model of convex function \( f \).

Note

1. \( f_t(x) \leq f(x), \forall x \in X \)
2. \( f_t(x_i) = f(x_i), \forall 1 \leq i \leq t \)
3. \( f_1(x) \leq f_2(x) \leq \ldots \leq f_t(x) \leq \ldots \leq f(X) \)

25.2.1 Kelley method (Kelley, 1960)

The Kelley method works as follows:

\[ x_{t+1} = \arg \min_{x \in X} f_t(x) \]

Obviously, the above algorithm converges so long as \( X \) is compact. The auxiliary problem is not so disturbing (reduces to LP) when \( X \) is polyhedron. However, the issue is that \( x_t \) is not unique and Kelley method can be very unstable. Indeed, the worst-case complexity of Kelley method is at least \( O\left(\frac{1}{\epsilon^{m-n+1}}\right) \).

Remedy: To prevent the instability issue, a possible remedy is update \( x_{t+1} \) by

\[ x_{t+1} = \arg \min_{x \in X} \left\{ f_t(x) + \frac{\alpha_t}{2} \| x - x_t \|_2^2 \right\} \]

with properly selected \( \alpha_t > 0 \).

25.2.2 Level Method (Lemarchal, Nemirovski, Nesterov, 1995)

Denote

\[ f_t = \min_{x \in X} f_t(x) \quad \text{(minimal value of the model)} \]
\[ \tilde{f}_t = \min_{1 \leq i \leq t} f(x_i) \quad \text{(record value of the model)} \]

we have \( f_1 \leq f_2 \leq \ldots \leq f^* \leq \ldots \leq \tilde{f}_2 \leq \tilde{f}_1 \)

Denote the level set

\[ L_t = \{ x : f_t(x) \leq l_t := (1 - \alpha) f_t + \alpha \tilde{f}_t \} \]

Note that \( L_t \) is nonempty, convex and closed, and doesn’t contain the search points \( \{x_1, \ldots, x_t\} \)
The Level method works as follows

\[ x_{t+1} = \Pi_{L_t}(x_t) = \arg \min_{x \in X} \left\{ \| x - x_t \|_2^2 : f_t(x) \leq l_t \right\} \]

Note when \( \alpha = 0 \), reduces to Kelley method. \( \alpha = 1 \), there will be no progress.

The auxiliary problem reduces to a quadratic program when \( X \) is polyhedron.

**Theorem 25.6** When \( t > \frac{1}{(1-\alpha)^2\alpha(2-\alpha)} \left( \frac{M_fD_X}{\epsilon} \right)^2 \), we have

\[ \bar{f}_t - \bar{f}_t \leq \epsilon \]

where \( M_f \) is the Lipschitz constant and \( D_X \) is the diameter of set \( X \).

**Remark:** Level method achieves same complexity as the subgradient method (which indeed is unimprovable), but can perform much better than subgradient method in practice.